# Algebraic Approach to the $1/N$ Expansion in Quantum Field Theory

Stefan Hollands*
*Enrico Fermi Institute, Department of Physics,*
*University of Chicago, 5640 Ellis Ave.,*
*Chicago IL 60637, USA*

February 7, 2008

**Abstract**

The $1/N$ expansion in quantum field theory is formulated within an algebraic framework. For a scalar field taking values in the $N$ by $N$ hermitian matrices, we rigorously construct the gauge invariant interacting quantum field operators in the sense of power series in $1/N$ and the 't Hooft coupling parameter as members of an abstract *-algebra. The key advantages of our algebraic formulation over the usual formulation of the $1/N$ expansion in terms of Green's functions are (i) that it is completely local so that infra-red divergencies in massless theories are avoided on the algebraic level and (ii) that it admits a generalization to quantum field theories on globally hypberbolic Lorentzian curved spacetimes. We expect that our constructions are also applicable in models possessing local gauge invariance such as Yang-Mills theories.

The $1/N$ expansion of the renormalization group flow is constructed on the algebraic level via a family of *-isomorphisms between the algebras of interacting field observables corresponding to different scales. We also consider $k$-parameter deformations of the interacting field algebras that arise from reducing the symmetry group of the model to a diagonal subgroup with $k$ factors. These parameters smoothly interpolate between situations of different symmetry.

## 1 Introduction

A common strategy to gain (at least approximate) information about physical models is to expand quantities of interest in terms of the parameters of the model. For example

*Electronic mail: `stefan@bert.uchicago.edu`

in perturbation theory, one expands in terms of the coupling parameter(s) of the theory. In quantum theories, it is sometimes fruitful to expand in terms of Planck's constant, $\hbar$. The key point in all cases is that the theory one is expanding about—often a linear theory in the first example, and the classical limit in the second example—is under better control, and that there exist, in many cases, systematic and constructive schemes to calculate the deviations order by order. Another such expansion that has by now become standard in quantum field theory is the expansion in $1/N$, where $N$ describes the number of components of the field(s) in the model. As in the previous two examples, the theory that one expands about, i.e., the large $N$ limit, is often somewhat simpler than the theory at finite $N$, and can sometimes even be solved exactly. (Just as an example, it can happen [17] that the large $N$ limit of a non-renormalizable theory is renormalizable, and the $1/N$-corrections remain renormalizable.)

The $1/N$ expansion in quantum field theory was first introduced by 't Hooft [15] in the context of non-abelian gauge theories. He observed that, if one explicitly keeps track of all factors of $N$ in the perturbative expansion of a connected Green's function of gauge invariant interacting fields, then the series can be organized as a power series in $1/N$, provided that the coupling parameter of the theory is also chosen to depend on $N$ in a suitable way. Moreover, he showed that the Feynman diagrams associated with terms at a given order in $1/N$ can naturally be related to Riemann surfaces with a number of handles equal to that order. Thus, in the $1/N$ expansion of this model, the leading contribution corresponds to planar diagrams, the subleading contribution to diagrams with a toroidal topology, etc.

The usual schemes for calculating the Green's functions in perturbation theory implicitly assume that the interacting fields approach suitable "in"-fields in the asymptotic past, which one assumes can be identified with the fields in the underlying free field theory that one is expanding about. The existence of such "in"-fields is closely related with the possibility to interpret the theory in terms of particles, and with the existence of an $S$-matrix. However, none of these usually exist in massless theories. Thus, the formulation of the $1/N$ expansion in terms of Green's functions is potentially problematical in massless theories. These problems come into even sharper focus if one considers theories on non-static (globally hypberbolic) Lorentzian spacetimes. Here there is not, in general, available even a preferred vacuum state based on which to calculate the Green's functions. Moreover, one would certainly not expect the fields to approach free "in"-fields in the asymptotic past for example in spacetimes that do not have suitable static regions in the asymptotic future and past (such as our very own universe).

A strategy to avoid these difficulties has recently been developed in [3, 4]. The new idea in those references is to construct directly the interacting field *operators* as members of some *-algebra of observables, rather than trying to construct the Green's functions. The key advantage of this approach is that, as it turns out, the interacting fields operators can always be defined in a completely satisfactory way, without any reference to an imagined (in general non-existent) "in"-field, or "in"-states in the asymptotic past. Consequently,

infra-red problems do not arise on the level of the interacting field observables and their associated algebras[1]. Not surprisingly, these ideas have also been a key ingredient in the construction of interacting quantum field theories in curved spacetimes [11, 12, 10].

The purpose of this article is to show that it is also possible to formulate the $1/N$ expansion directly in terms of the interacting fields and their associated algebras of observables to which these fields belong, thereby achieving a complete disentaglement from the infra-red behavior of the theory. We will consider in this article explicitly only the theory of a scalar field in the $\mathbf{N} \otimes \bar{\mathbf{N}}$ representation of $U(N)$ on Minkowski space. However, since our algebraic construction is done without making use of any of the particlular features of Minkowski space, it can be generalized to arbitrary Lorentzian curved spacetimes by the methods of [3, 11, 12]. Also, we expect that our methods are applicable to models with *local* gauge symmetry such as Yang-Mills theories, although the algebraic construction of the interacting fields in such theories is more complex due to the presence of unphysical degrees of freedom, and still a subject of investigation, see e.g. [5, 16, 7].

We now summarize the contents of this paper. In section 2 we review, in a pedagogical way, the construction of the field observables and the corresponding algebra associated with a single, free hermitan scalar field $\phi$. This algebra is sufficiently large to contain the Wick powers of $\phi$ and their time ordered products, which are required later in the construction of the corresponding interacting quantum field theory. A description of the properties of these objects and their construction is therefore included.

In section 3, we generalize these constructions to a free scalar field in the $\mathbf{N} \otimes \bar{\mathbf{N}}$ representation of $U(N)$, and we show how to construct the $1/N$ expansion of this theory on the level of field observables and their associated algebras.

In section 4, we proceed to interacting quantum field theories. Based on the algebraic construction of the underlying free quantum field theory in section 3, we construct the interacting quantum fields as formal power series in $1/N$ and the 't Hooft coupling as members of a suitable algebra $\mathcal{A}_V$, where $V$ is a gauge invariant interaction. The contributions to these quantities arising at order $H$ in the $1/N$ expansion correspond precisely to Feynman diagrams whose toplogy is that of a Riemann surface with $H$ handles. We point out that the algebras $\mathcal{A}_V$ also incorporate an expansion in $\hbar$ ("loop-expansion") [4], as well as by construction an expansion the coupling parameters appearing in $V$. Therefore, the construction of $\mathcal{A}_V$ in fact incorporates all three expansions mentioned at the beginning.

In section 5 we review, first for a single scalar field, the formulation of the renormalization group in the algebraic framework [10] that we are working in. We then show that this construction can be generalized to an interacting field in the $\mathbf{N} \otimes \bar{\mathbf{N}}$ representation of $U(N)$ with gauge invariant interaction, defined in the sense of a power series in $1/N$. The renormalization group map, is therefore also defined as a power series in $1/N$.

[1]Such divergences will arise, if one tries to construct (non-existent) quantum states in the theory corresponding to "free incoming particles".

In section 6, we vary the constructions of sections 3 and 4 by considering interactions that are invariant only under some subgroup of $U(N)$. We show that, for a diagonal subgroup with $k$ factors, the algebraic construction of the $1/N$ expansion can still be carried through, but now leads to deformed algebars of interacting field observables which are labeled by $k$ real deformation parameters associated with the relative size of the subgroups. These parameters smoothly interpolate between situations of different symmetry. The contributions to an interacting field at order $H$ in $1/N$ are now associated with Riemann surfaces that are "colored" by $k$ "spins", where each coloring is weighted according to the values of the deformation parameters.

# 2 Algebraic construction of a single scalar quantum field

The perturbative construction of an interacting quantum field theory is based on the construction on the corresponding free quantum field theory, and we shall therefore begin by considering free fields. In this section, we will review how to define an algebra of observables associated with a single free hermitian Klein-Gordon field of mass $m$, described by the the classical action[2]

$$S = \int (\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2) \, d^d x, \tag{1}$$

which is large enough in order to contain the Wick powers and the time ordered products of the field $\phi$. Our review is essentially self-contained and follows the ideas developed in [4, 10, 11, 12, 3], which the reader may look up for details.

For pedagogical purposes, we begin by defining first a "minimal algebra" of observables associated with the action (1). Consider the free *-algebra over the complex numbers generated by a unit $\mathbf{1}$ and formal expressions $\phi(f)$ and $\phi(h)^*$, where $f$ and $h$ run through the space of compactly supported smooth testfunctions on $\mathbb{R}^d$. The minimal algebra is obtained by factoring this free algebra by the following relations.

1. (linearity) $\phi(af + bh) = a\phi(f) + b\phi(h)$ for all $a, b \in \mathbb{C}$ and testfunctions $f, h$.
2. (field equation) $\phi((\partial^\mu \partial_\mu - m^2) f) = 0$ for all testfunctions $f$.
3. (hermiticity) $\phi(f)^* = \phi(\bar{f})$.
4. (commutation relations) $[\phi(f), \phi(h)] = i\Delta(f, h) \cdot \mathbf{1}$, where $\Delta(f, h)$ is the advanced minus retarded propagator for the Klein-Gordon equation, smeared with the testfunctions $f$ and $h$.

[2]Our signature convention is $- + + + \ldots$.

We formally think of the expressions $\phi(f)$ as the "smeared" quantum fields, i.e., the integral of the formal[3] pointlike quantum field against the test function $f$,

$$\phi(f) = \int_{\mathbb{R}^d} \phi(x) f(x)\, d^d x. \tag{2}$$

The linearity of the expression $\phi(f)$ in $f$ corresponds to the linearity of the integral. Relation 2) is the field equation for $\phi(x)$ in the sense of distributions, i.e., it formally corresponds to the Klein-Gordon equation for $\phi(x)$ via a partial integration. Relation 3) says that the field $\phi$ is hermitian and relation 4) implements the usual commutation relations of the free hermitian scalar field on $d$-dimensional Minkowski spacetime.

The minimal algebra is too small for our purposes. It does not, for example, contain observables corresponding to Wick powers of the field $\phi$ at the same spacetime point, nor their time ordered products. These are, however, required if one wants to construct the interacting quantum field theory perturbatively around the free theory. We now construct an enlarged algebra, $\mathcal{W}$, which contains elements corresponding to these observables. For this purpose, it is useful to first present the minimal algebra in terms of a new set of generators, defined by $W_0 = \mathbf{1}$, and

$$W_a(f_1 \otimes \cdots \otimes f_a) = (-i)^a \frac{\partial^a}{\partial \lambda_1 \cdots \partial \lambda_a} \mathrm{e}^{i\phi(F)} \mathrm{e}^{\frac{1}{2}\Delta_+(F,F)} \bigg|_{\lambda_i = 0}, \quad F = \sum^a \lambda_i f_i, \tag{3}$$

where $\Delta_+$ is any distribution in 2 spacetime variables which is a solution to the Klein-Gordon equation in each variable, and which has the property that its antisymmetric part is equal to $(i/2)\Delta$. In particular, we could choose $\Delta_+ = (i/2)\Delta$ at this stage, but it is important to leave this choice open for later. It follows from the definition that $W_1(f) = \phi(f)$, that the quantities $W_a$ are symmetric under exchange of the testfunctions, and that

$$W_a(f_1 \otimes \cdots \otimes f_a)^* = W_a(\bar{f}_1 \otimes \cdots \otimes f_a), \quad W_a(f_1 \otimes \cdots (\partial^\mu \partial_\mu - m^2) f_i \otimes \cdots f_a) = 0. \tag{4}$$

Using the algebraic relations 1)–4), one can express the product of two such quantities again as a linear combination of such quantities,

$$\begin{aligned} &W_a(f_1 \otimes \cdots \otimes f_a) \cdot W_b(h_1 \otimes \cdots \otimes h_b) \\ &\qquad = \sum_{\mathcal{P}} \prod_{(k,l) \in \mathcal{P}} \Delta_+(f_k, h_l) \cdot W_c(\otimes_{i \notin \mathcal{P}_1} f_i \otimes \otimes_{j \notin \mathcal{P}_2} h_j), \end{aligned} \tag{5}$$

where the following notation has been used to organize the sum on the right side: We consider sets $\mathcal{P}$ of pairs $(i,j) \in \{1, \dots, a\} \times \{1, \dots, b\}$; we say that $i \notin \mathcal{P}_1$ if there is no $j$

[3] Since $\Delta$ is a distribution, relation 4) implies that the field $\phi$ necessarily has a distributional character. Therefore, the field only makes good mathematical sense after smearing with a test function.

such that $(i,j) \in \mathcal{P}$, and we say $j \notin \mathcal{P}_2$ if there is no $i$ such that $(i,j) \in \mathcal{P}$. The number $c$ is related to $a$ and $b$ by $a + b - c = 2|\mathcal{P}|$, where $|\mathcal{P}|$ is the number of pairs in $\mathcal{P}$.

It is easy to see that relations (4) and (5) form an equivalent presentation of the minimal algebra, i.e., we could equivalently *define* the minimal algebra to be the abstract algebra generated by the elements $W_a(\otimes_i f_i)$, subject to the relations (4) and (5), instead of defining it as the algebra generated by $\phi(f)$ subject to the relations 1)–4) above (this is true no matter what the particular choice of $\Delta_+$ is). Thus, all we have done so far is to rewrite the minimal algebra in terms of different gerenators.

To obtain the desired extension, $\mathcal{W}$, of the minimal algebra, we now choose a distribution $\Delta_+$ which is not only a bisolution to the Klein-Gordon equation with the property that its antisymmetric part is equal to $(i/2)\Delta$, but which has the additional property that that it is of positive frequency type in the first variable and of negative frequency type in the second variable. This condition is formalized by demanding that the wave front set[4] $\mathrm{WF}(\Delta_+)$, (for the definition of the wave front set of a distribution, see [9]) has the following, so-called "Hadamard", property

$$\begin{aligned}\mathrm{WF}(\Delta_+) \subset \{(x_1, x_2; p_1, p_2) \in (\mathbb{R}^d \times \mathbb{R}^d) \times (\mathbb{R}^d \times \mathbb{R}^d \setminus (0,0)) \mid \\ p_1 = -p_2, (x_1 - x_2)^2 = 0, p_1 \in \bar{V}^+\} \equiv \mathcal{C}_+, \quad (6)\end{aligned}$$

where $\bar{V}^{\pm}$ denote the closure of the future resp. past lightcone in $\mathbb{R}^d$. The key point is now that the relations (4) and (5) make sense not only for test *functions* of the form $f_1 \otimes \cdots \otimes f_a$, but even much more generally for any test *distribution*, $t$, in the space $\mathcal{E}'_a$ of compactly supported distributions in $a$ spacetime arguments which have the property that their wave front set does not contain any element of the form $(x_1, \dots, x_n; p_1, \dots, p_n)$ such that all $p_i$ are either in the closure of the forward lightcone, or the closure of the past light cone,

$$\begin{aligned}\mathcal{E}'_a = \{t \in \mathscr{D}'(\times^a \mathbb{R}^d) \mid t \text{ comp. supp., } \mathrm{WF}(t) \text{ has no element} \\ \text{in common with } (\times^a \mathbb{R}^d) \times (\times^a \bar{V}^+) \text{ or } (\times^a \mathbb{R}^d) \times (\times^a \bar{V}^-)\}. \quad (7)\end{aligned}$$

Indeed, these conditions on the wave front set on the $t$, together with the wave front set properties (6) of $\Delta_+$ can be shown to guarantee that the potentially ill defined products of distributions occurring in a product $W_a(t) \cdot W_b(s)$ are in fact well-defined and are such that the resulting terms are each of the form $W_c(u)$, with $u$ again an element in the space $\mathcal{E}'_c$.[5]

We take $\mathcal{W}$ to be the algebra generated by symbols of the form $W_a(t)$, $t \in \mathcal{E}'_a$, subject to the relations (4) and (5), with $\otimes_i f_i$ in those relations replaced by distributions in the

[4]It can be shown that the wave front set of a distribution is actually invariantly defined as a subset of the cotangent space of the manifold on which the distribution is defined. The set (6) should therefore be intrinsically thought of as a subset of $T^*(\mathbb{R}^d \times \mathbb{R}^d)$.

[5]We note that the wave front set of $(i/2)\Delta$ is *not* of Hadamard type, and $\Delta_+ = (i/2)\Delta$ is not a possible choice in the construction of $\mathcal{W}$.

spaces $\mathcal{E}'_a$. Our definition of the generators $W_a(t)$ depends on the particular choice of $\Delta_+$. However, as an abstract algebra, $\mathcal{W}$ is independent of this choice [11]. To see this, choose any other bidistribution $\Delta'_+$ with the same wave front set property as $\Delta_+$, and let $\mathcal{W}'$ be the corresponding algebra with generators $W'_a(t)$ defined as in eq. (3). Then $\mathcal{W}$ and $\mathcal{W}'$ are isomorphic. The isomorphism is given in terms of the generators

$$W_a(t) \to \sum_{2n \le a} \frac{a!}{(2n)!(a-2n)!} W'_{a-2n}(\langle F^{\otimes n}, t\rangle), \tag{8}$$

where $F = \Delta_+ - \Delta'_+$, and where $\langle F^{\otimes n}, t\rangle$ is the compactly supported distribution on $\times^{a-2n}\mathbb{R}^d$ defined by

$$\langle F^{\otimes n}, t\rangle(y_1, \dots, y_{a-2n}) = \int t(x_1, \dots, x_{2n}, y_1, \dots, y_{a-2n}) \prod_i F(x_i, x_{i+1}) \prod d^d x_i. \tag{9}$$

(That this distribution is in the class $\mathcal{E}'_{a-2n}$ follows from the wave front set properties of $\Delta_+, \Delta'_+$, which, together with the wave equation imply that $F$ is smooth.)

This completes our construction of the algebra of quantum observables for a single Klein-Gordon field associated with the action (1). Quantum *states* in the algebraic framework are by definition linear functionals $\omega : \mathcal{W} \to \mathbb{C}$ which are positive in the sense that $\omega(A^*A) \ge 0$ for all $A \in \mathcal{W}$, and which are normalized so that $\omega(\mathbf{1}) = 1$. This algebraic notion of a quantum state encompasses the usual Hilbert-space notion of state, in the sense that any vector or density matrix in a Hilbert-space on which the elements of $\mathcal{W}$ are represented as linear operators defines an algebraic state in the above sense via taking expectation values. Conversely, given an algebraic state $\omega$, the GNS-construction yields a representation $\pi$ on a Hilbert space $\mathcal{H}$ containing a vector $|\Omega\rangle$ such that $\omega(A) = \langle\Omega|\pi(A)|\Omega\rangle$. Note, however, that it is not true that any state on $\mathcal{W}$ arises in this way from a *single*, given Hilbert-space representation (this is closely related to the fact that $\mathcal{W}$ has (many) inequivalent representations). The algebra $\mathcal{W}$ can be equipped with a unique topology that makes the product and *-operation continuous [11], and the notion of a *continuous* state on $\mathcal{W}$ can thereby be defined. The continuous states $\omega$ on $\mathcal{W}$ can be characterized entirely in terms of the $n$-point distributions $\omega(\phi(x_1)\cdots\phi(x_n))$ of the field $\phi$. Moreover, it can be shown [14], that the continous states are precisely those for which the 2-point distribution has wave front set eq. (6), and for which the so-called "connected" $n$-point distributions are smooth for $n \neq 2$.

The invariance of the action (1) under the Poincare-group is reflected in a corresponding invariance of $\mathcal{W}$, in the sense that $\mathcal{W}$ admits an automorphic action of the Poincare-group: For any element $\{\Lambda, a\}$ of the Poincare group consisting of a proper, orthochronous Lorentz transformation $\Lambda$ and a translation vector $a \in \mathbb{R}^d$, there is an automorphism $\alpha_{\{\Lambda,a\}}$ on $\mathcal{W}$ satisfying the composition law $\alpha_{\{\Lambda,a\}} \circ \alpha_{\{\Lambda',a'\}} = \alpha_{\{\Lambda,a\}\cdot\{\Lambda',a'\}}$. The action of this automorphism is most easily described if we choose a $\Delta_+$ which is invariant under the Poincare-group. (Since $\mathcal{W}$ is independent of the choice of $\Delta_+$, we may

do so if we like.) An admissible[6] choice for $\Delta_+$ with the above properties is the Wightman function of the free field,

$$\Delta_+(x,y) = w^{(m)}(x-y) \equiv \frac{1}{(2\pi)^{d-1}} \int_{p^0 \geq 0} \delta(p^2 - m^2) e^{ip(x-y)}\, d^d p. \tag{10}$$

With this choice, the action of $\alpha_{\{\Lambda,a\}}$ is simply given by[7]

$$\alpha_{\{\Lambda,a\}}(W_a(t)) = W_a(t \circ \{\Lambda, a\}). \tag{11}$$

Furthermore, with this choice for $\Delta_+$, the generators $W_a(t)$ correspond to the usually considered normal ordered products of fields,

$$W_a(f_1 \otimes \cdots \otimes f_a) = :\prod_{i=1}^{a} \phi(f_i):, \tag{12}$$

and the product formula (5) simply corresponds to "Wick's theorem" for multiplying to Wick-polynomials.

Since $W_1(f) = \phi(f)$, the enlarged algebra $\mathcal{W}$ contains the minimal algebra generated by the free field $\phi(f)$ as a subalgebra. In fact, $\mathcal{W}$ also contains Wick powers of the field at the same spacetime point as well as their time-ordered products, which are not in the minimal algebra. These objects can be characterized axiomatically (not uniquely, as we shall see) by a number of properties that we will list now. In order to state these properties of in a convenient way, let us introduce the vector space $\mathcal{V}$ whose basis elements are labelled by formal products of the field $\phi$ and its derivatives,

$$\mathcal{V} = \text{span}\{\mathcal{O} = \prod \partial_{\mu_1} \cdots \partial_{\mu_k} \phi\}, \tag{13}$$

so that each element of $\mathcal{V}$ is given by a formal linear combination $\sum g_j \mathcal{O}_j$, $g_j \in \mathbb{C}$. We refer to the elements of $\mathcal{V}$ as "formal" field expressions, because no relations such as the field equation are assumed to hold at this stage. Consider, furthermore, the space $\mathscr{D}(\mathbb{R}^d; \mathcal{V})$ of smooth functions of compact support whose values are elements in the vector space $\mathcal{V}$. Thus, any element $F \in \mathscr{D}(\mathbb{R}^d; \mathcal{V})$ can be written in the form $F(x) = \sum f_i(x) \mathcal{O}_i$, where $\mathcal{O}_i$ are basis elements in $\mathcal{V}$, and where $f_i$ are complex valued smooth functions on $\mathbb{R}^d$ of compact support.

We view the Wick powers as linear maps

$$\mathscr{D}(\mathbb{R}^d; \mathcal{V}) \to \mathcal{W}, \quad f\mathcal{O} \to \mathcal{O}(f) \tag{14}$$

[6] That the wave front set of the Wightman function is equal to (6) is proved e.g. in [18].

[7] That $t \circ \{\Lambda, a\}$ is again an element of $\mathcal{E}'_a$ is a consequence of the covariant transformation law $\mathrm{WF}(f^* t) = f^* \mathrm{WF}(t)$ of the wave front set [9], where $f$ can be any diffeomorphism, together with the fact that the future/past lightcones are preserved under the action of the proper, orthochronous Poincare group. On the other hand, a Lorentz transformation reversing the time orientation does not preserve the spaces $\mathcal{E}'_a$ and consequently does not give rise to an automorphism of $\mathcal{W}$.

and the $n$-fold time ordered products as multi linear maps

$$T : \times^n \mathscr{D}(\mathbb{R}^d; \mathcal{V}) \to \mathcal{W}, \quad (f_1\mathcal{O}_1, \dots, f_n\mathcal{O}_n) \to T(\prod f_i\mathcal{O}_i). \tag{15}$$

Time ordered products with only one factor are required to be given by the corresponding Wick power,

$$T(f\mathcal{O}) \equiv \mathcal{O}(f). \tag{16}$$

The further properties required from the time ordered products (including the Wick powers as a special case) are the following[8]:

(t1) (symmetry) The time ordered products are symmetric under exchange of the arguments.

(t2) (causal factorization) If $I$ is a subset of $\{1, \dots, n\}$, and if the supports of $\{f_i\}_{i\in I}$ are in the causal future of the supports of $\{f_j\}_{j\in I^c}$ ($I^c$ denotes the complement of $I$), then we ask that

$$T\left(\prod_i f_i\mathcal{O}_i\right) = T\left(\prod_{i\in I} f_i\mathcal{O}_i\right) T\left(\prod_{j\in I^c} f_j\mathcal{O}_j\right). \tag{17}$$

(t3) (commutator)

$$\begin{aligned}\left[T\left(\prod_{i=1}^n f_i\mathcal{O}_i\right), \phi(h)\right]& \\ = i\sum_k T&\left(f_1\mathcal{O}_1 \cdots \sum_{\mu_1\dots\mu_l} (h\partial_{\mu_1}\dots\partial_{\mu_l}\Delta * f_k)\frac{\partial\mathcal{O}_k}{\partial(\partial_{\mu_1}\dots\partial_{\mu_l}\phi)} \cdots f_n\mathcal{O}_n\right),\end{aligned} \tag{18}$$

where we have set $(\Delta * f)(x) = \int \Delta(x-y)f(y)\, d^dy$.

(t4) (covariance) Let $\{\Lambda, a\}$ be a Poincare transformation. Then

$$\alpha_{\{\Lambda,a\}}\left(T(\prod_i f_i\mathcal{O}_i)\right) = T\left(\prod_i \psi^*_{\{\Lambda,a\}}(f_i\mathcal{O}_i)\right), \tag{19}$$

where $\psi^*_{\{\Lambda,a\}}$ denotes the pull-back of an element in $\mathscr{D}(\mathbb{R}^d; \mathcal{V})$ by the linear transformation $x \to \Lambda x + a$.

[8]Actually, one ought to impose additional renormalization conditions specifically for time ordered products containing derivatives of the fields, see e.g. [5] and [13], beyond the requirements (t1)–(t8) below. Such conditions are important e.g. in order to show that the field equations or conservation equations hold for the interacting fields, but they do not play a role in the present paper. We have therefore omitted them here to keep things as simple as possible.

(t5) (scaling) The time ordered products have the following "almost homogeneous" scaling behavior under simultaneous rescalings of the intertial coordinates and the mass, $m$. Let $\lambda > 0$, and set $f^\lambda(x) = \lambda^{-n} f(\lambda x)$ for any function or distribution on $\mathbb{R}^n$. For a given prescription $T^{(m)}$ for the value $m$ of the mass (valued in the algebra $\mathcal{W}^{(m)}$ associated with this value of the mass), consider the new prescription $T^{(m)\prime}$ defined by

$$T^{(m)\prime}\left(\prod_i f_i \mathcal{O}_i\right) \equiv \lambda^{-\sum d_i} \sigma_\lambda \left[T^{(\lambda m)}\left(\prod_i f_i^\lambda \mathcal{O}_i\right)\right], \tag{20}$$

where $\sigma_\lambda : \mathcal{W}^{(\lambda m)} \to \mathcal{W}^{(m)}$ is the canonical isomorphism[9], and where $d_i$ is the "engineering dimension"[10] of the field $\mathcal{O}_i$. (Note that $T^{(\lambda m)}$ is valued in $\mathcal{W}^{(\lambda m)}$.) Then we demand that $T^{(m)\prime}$ depends at most logarithmically on $\lambda$ in the sense that[11]

$$T^{(m)\prime} = T^{(m)} + \text{polynomial expressions in } \ln \lambda. \tag{21}$$

(t6) (microlocal spectrum condition) Let $\omega$ be a continuous state on $\mathcal{W}$. Then the distributions $\omega_T : (f_1, \dots, f_n) \to \omega(T(\prod f_i \mathcal{O}_i))$ are demanded to have wave front set

$$\mathrm{WF}(\omega_T) \subset \mathcal{C}_T, \tag{22}$$

where the set $\mathcal{C}_T \subset (\times^n \mathbb{R}^d) \times (\times^n \mathbb{R}^d \setminus \{0\})$ is described as follows (we use the graphological notation introduced in [2, 3]): Let $\Gamma(p)$ be a "decorated embedded Feynman graph" in $\mathbb{R}^d$. By this we mean an embedded Feynman graph $\mathbb{R}^d$ whose vertices are points $x_1, \dots, x_n$ with valence specified by the fields $\mathcal{O}_i$ occurring in the time ordered product under consideration, and whose edges, $e$, are oriented null-lines [i.e., $(x_i - x_j)^2 = 0$ if $x_i$ and $x_j$ are connected by an edge]. Each such null line is equipped with a momentum vector $p_e$ parallel to that line. If $e$ is an edge in $\Gamma(p)$ connecting the points $x_i$ and $x_j$ with $i < j$, then $s(e) = i$ is its source and $t(e) = j$ its target. It is required that $p_e$ is future/past directed if $x_{s(e)}$ is not in the past/future of $x_{t(e)}$. With this notation, we define

$$\begin{aligned} \mathcal{C}_T \;=\; & \Big\{(x_1, \dots, x_n; k_1, \dots, k_n) \mid \exists \text{ decorated Feynman graph } \Gamma(p) \text{ with vertices} \\ & x_1, \dots, x_n \text{ such that } k_i = \sum_{e:s(e)=i} p_e - \sum_{e:t(e)=i} p_e \quad \forall i \Big\}. \end{aligned} \tag{23}$$

---

[9]The canonical isomorphism is defined by $\sigma_\lambda : \mathcal{W}^{(\lambda m)} \ni W_a(t) \to \lambda^{-a} \cdot W_a(t^\lambda) \in \mathcal{W}^{(m)}$.

[10]In scalar field theory in $d$ spacetime dimensions, the mass dimension of a field $\mathcal{O}$ is defined as the number of derivatives plus $(d-2)/2$ times the number of factors of $\phi$ plus twice the number of factors of $m^2$.

[11]The difference $T^{(m)} - T^{(m)\prime}$ describes the failiure of $T_m$ to scale *exactly* homogeneously. For the time ordered products with only one factor (i.e., the Wick powers), it can be shown that this difference vanishes, i.e., the Wick powers scale exactly homogeneously in Minkowski space. For the time ordered products with more than one factor, the logarithms cannot in general be avoided.

(t7) (unitarity) We have $T^* = \bar{T}$, where $\bar{T}$ is the "anti-time-ordered product, defined as

$$\bar{T}(f_1\mathcal{O}_1 \dots f_n\mathcal{O}_n) = \sum_{I_1 \sqcup \dots \sqcup I_j = \{1,\dots,n\}} (-1)^{n+j} T(\prod_{i\in I_1} \bar{f}_i \mathcal{O}_i) \dots T(\prod_{i\in I_j} \bar{f}_i \mathcal{O}_i), \tag{24}$$

where the sum runs over all partitions of the set $\{1, \dots, n\}$ into disjoint subsets $I_1, \dots, I_j$.

(t8) (smooth dependence upon $m$) The time ordered products depend smoothly upon the mass parameter $m$ in the following sense. Let $\omega^{(m)}$ be a 1-parameter family of states on $\mathcal{W}^{(m)}$. We say that $\omega^{(m)}$ depends smoothly upon $m$ if (1) the 2-point function $\omega_2^{(m)}(x, y)$ when viewed as a distribution jointly in $m, x, y$ has wave front set

$$\mathrm{WF}(\omega_2^{(m)}) \subset \{(x_1, x_2, m; p_1, p_2, \rho) \mid (p_1, p_2, \rho) \neq 0, (x_1, x_2; p_1, p_2) \in \mathcal{C}_+\}, \tag{25}$$

where the set $\mathcal{C}_+$ was defined above in eq. (6), and if (2) the truncated $n$-point functions $\omega_n^{(m)\mathrm{conn}}$ are smooth jointly in $m, x_1, \dots, x_n$. We say the prescription $T^{(m)}$ is smooth in $m$ if $\omega_T^{(m)}(f_1, \dots, f_n) = \omega^{(m)}(T^{(m)}(\prod f_i \mathcal{O}_i))$ (viewed as a distribution jointly in $m$ and its spacetime arguments) has wave front set

$$\mathrm{WF}(\omega_T^{(m)}) \subset \Big\{(x_1, \dots, x_n, m; k_1, \dots, k_n, \rho) \mid \\ (k_1, \dots, k_n, \rho) \neq 0, (x_1, \dots, x_n; k_1, \dots, k_n) \in \mathcal{C}_T\Big\} \tag{26}$$

for such a smooth family of states, where the set $\mathcal{C}_T$ was defined above in eq. (23).

It is relatively straightforward to demonstrate the existence of a prescription for defining the Wick powers as elements of $\mathcal{W}$ satisfying the above properties. For example, for the fields $\phi^a \in \mathcal{V}$, $a = 1, 2, \dots$, the corresponding algebra elements $\phi^a(f) \in \mathcal{W}$ satisfying the above properties may be defined as follows. Let $H^{(m)}(x, y)$ be any family of bidistributions satisfying the wave equation in both entries and the wave front set condition (6), whose antisymmetric part is equal to $(i/2)\Delta(x, y)$, and which has a smooth dependence upon $m$ in the sense of (t8). Define

$$\phi^a(x) = \frac{\delta^n}{i^n \delta f(x)^n} e^{i\phi(f) + \frac{1}{2}H^{(m)}(f,f)} \Bigg|_{f=0}. \tag{27}$$

Then $\phi^a(f) \in \mathcal{W}$ satisfies (t1)–(t8). This definition of $\phi^a(f)$ can be restated equivalently as follows: We may use the bidistribution $\Delta_+ = H^{(m)}$ in the definition of the generators $W_a$ (see eq. (3)) and the algebra product (5) of $\mathcal{W}$, since we have already argued that $\mathcal{W}$ is independent of the particular choice of $\Delta_+$. Consider the distribution $t$ given by

$$t(x_1, \dots, x_a) = f(x_1)\delta(x_1 - x_2) \cdots \delta(x_{a-1} - x_a), \tag{28}$$

where $\delta$ is the ordinary delta-distribution in $\mathbb{R}^d$. Then one can show that $t$ is in the class of distributions $\mathcal{E}'_a$, and definition (27) (in smeared form) is equivalent to setting

$$\phi^a(f) = W_a(t) \in \mathcal{W}, \tag{29}$$

where it is understood that $W_a$ is defined in terms of $\Delta_+ = H^{(m)}$. Wick powers containing derivatives are defined in a similiar way via suitable derivatives of delta distributions.

The usual "normal ordering" prescription for Wick powers would correspond to setting $H^{(m)}$ equal to the Wightman 2-point function $w^{(m)}$ given above in eq. (10). However, this is actually not an admissible choice in our framework since, by inspection $w^{(m)}$ (and hence the vacuum state) does *not* depend smoothly upon $m$ in the sense of (t8). In fact, the Wightman 2-point function $w^{(m)}(x-y)$ explicitly contains a term of the form $J[m^2(x-y)^2]\log m^2$ with a logarithmic dependence upon the mass $m$, where $J$ is a smooth (in fact, analytic) function that can be expressed in terms of Bessel functions. For this reason, the usual normal ordering prescription violates our condition (t8) that the Wick powers have a smooth dependence upon $m$. An admissible choice for $H^{(m)}$ is e.g. $w^{(m)}$ without this logarithmic term,

$$H^{(m)}(x, y) = w^{(m)}(x-y) - J[m^2(x-y)^2]\log m^2. \tag{30}$$

Since normal ordering is not admissible in our framework, it follows that no prescription for Wick powers satisfying (t1)–(t8) can have the property that it has a vanishing expectation value in the vacuum state for *all* values of $m \in \mathbb{R}$, because this property precisely distinguishes normal ordering. However, we can always adjust our prescription within the freedom left over by (t1)–(t8) in such a way that all Wick powers have a vanishing expectation value in the vacuum state for an arbitrary, but *fixed* value of $m$. It is therefore clear that, in practice, our prescription is just as viable as the usual normal ordering prescription, since $m$ can take on only one value. On the other hand, our prescription would lead to different predictions in a theory containing a spacetime dependent mass.

It is not possible to give a similarly explicit construction of time ordered products satisfying (t1)–(t8) with more than one factor. Using the ideas of "causal perturbation theory" (see e.g. [19]) one can, however, give an inductive construction of the time ordered products so that (t1)–(t8) are satisfied which is based upon the above construction of the Wick powers (i.e., time ordered products with one factor). These constructions are described in detail in [3, 5, 12] (see expecially [12] for the proof that scaling property (t5) can be satisfied), and we will therefore only sketch the key steps and ideas going into this inductive construction, referring the reader to the references for details.

The main idea behind the inductive construction is that the causal factorization property expressing the temporal ordering of the factors in the time ordered product already defines time ordered products with the desired properties for non-coinciding spacetime points once the Wick powers are known. Namely, if e.g. $\text{supp} f_1$ is before $\text{supp} f_2$, $\text{supp} f_2$

is before supp$f_3$ etc., then the causal factorization property tells us that we must have

$$T(\prod_i f_i \mathcal{O}_i) = \mathcal{O}_1(f_1) \cdots \mathcal{O}_n(f_n). \tag{31}$$

Since the Wick powers on the right side have already been constructed, we may take this relation as the definition of the time ordered products for testfunctions[12] $F = \otimes_i^n f_i$ whose support has no intersection with any of the "partial diagonals"

$$D_I = \{(x_1, \dots, x_n) \in \times^n \mathbb{R}^d \mid x_i = x_j \quad \forall i, j \in I\}, \quad I \subset \{1, \dots, n\}, \tag{32}$$

in the product manifold $\times^n \mathbb{R}^d$, because one can decompose such $F$ into contributions whose supports are temporally ordered via a partition of unity [3]. The causal factorization property alone therefore already defines the time ordered products as $\mathcal{W}$-valued distributions, denoted $T^0$, on the space $\times^n \mathbb{R}^d$, minus the union $\cup_I D_I$ of all partial diagonals, and it can furthermore be seen that these objects have the desired properties (t1)–(t8) on that domain. In order to define the time ordered products as distributions on all of $\times^n \mathbb{R}^d$, one has to construct a suitable extension $T$ of $T^0$ to a distribution defined on all of $\times^n \mathbb{R}^d$ in such a way that (t1)–(t8) are preserved in the extension process. This step corresponds to the usual "renormalization" step in other approaches and is the hard part of the analysis. Actually, we can even assume that $T^0$ is already defined everywhere apart from the *total* diagonal $D_n = \{x_i \neq x_j \ \forall i, j\}$, since one can construct the extension $T$ inductively in the number of factors. Having constructed these for up to less or equal than $n-1$ factors then leaves the time ordered products with $n$ factors undetermined only on the total diagonal.

A key simplification for the extension problem occurs because the commutator condition (inductively known to hold for $T^0$) can be shown to be equivalent to the following "Wick-expansion" for $T^0$,

$$T^0\left(\prod_{i=1}^n \mathcal{O}_i(x_i)\right) = \sum_{\alpha_1, \alpha_2, \dots} \frac{1}{\alpha_1! \dots \alpha_n!} \times \\ \tau^0\left[\delta^{\alpha_1}\mathcal{O}_1 \otimes \cdots \otimes \delta^{\alpha_n}\mathcal{O}_n\right](x_1, \dots, x_n) :\prod_{i=1}^n \prod_j [(\partial)^j \phi(x_i)]^{\alpha_{ij}} :_H . \tag{33}$$

Here, the $\tau^0[\otimes_i \Psi_i]$ are c-number distributions on $\times^n \mathbb{R}^d \setminus D_n$ [in fact equal to the expectation value of the time ordered product $T^0(\prod_i \Psi_i)$] depending in addition upon an arbitrary collection of fields $\Psi_1 \otimes \cdots \otimes \Psi_n \in \otimes^n \mathcal{V}$, each $\alpha_j$ is a multi index and we are using the notation

$$\delta^\alpha \mathcal{O} = \left\{ \prod_j \left( \frac{\partial}{\partial[(\partial)^j \phi]} \right)^{\alpha_j} \right\} \mathcal{O} \in \mathcal{V} \tag{34}$$

[12] Note that the time ordered products can be viewed as multi linear maps $\times^n \mathscr{D}(\mathbb{R}^d) \to \mathcal{W}$ for a fixed choice of fields $\mathcal{O}_1, \dots, \mathcal{O}_n$.

as well as $\alpha! = \prod_j \alpha_j!$ for multi indices[13]. The notation $: :_H$ stands for the "$H$-normal ordered products", defined by

$$: \prod_{i=1}^{k} \phi(x_i) :_H = \frac{\delta^k}{i^k \delta f(x_1) \dots \delta f(x_k)} e^{i\phi(f) + \frac{1}{2} H^{(m)}(f,f)} \Bigg|_{f=0} . \tag{35}$$

The key point about the Wick expansion is that it reduces the problem of extending the algebra valued $T^0$ to the problem of extending the c-number distributions $\tau^0$. Since we want the extensions $T$ to satisfy (t1)–(t8), we also want the extensions $\tau$ of the $\tau^0$ to satisfy a number of corresponding properties: First, the wave front set condition on the $T$ correspond to the requirement that $\mathrm{WF}(\tau) \subset \mathcal{C}_T$, where the set $\mathcal{C}_T$ was defined above in eq. (23). Second, since the $T$ are required to be Poincare invariant, also the extension $\tau$ must be Poincare invariant. Finally, since the $T$ are supposed to have an almost homogeneous scaling behavior under a rescaling $x \to \lambda x$ (and a simultaneous rescaling $m \to \lambda^{-1} m$ of the mass), the $\tau$ must have the scaling behavior

$$\left( \frac{\partial}{\partial \log \lambda} \right)^k \left\{ \lambda^D \tau^{(\lambda^{-1} m)}(\lambda x_1, \dots, \lambda x_n) \right\} = 0, \tag{36}$$

for some $k$, where $D$ is the sum of the mass dimensions of the fields $\Psi_i$ on which $\tau$ depends, and where we are indicating explicitly the dependence of $\tau$ upon the mass parameter[14]. By induction, these properties are already known for $\tau^0$ (i.e., off the total diagonal $D_n$), so the question is only whether they can also be satisfied in the extension process. To reduce this remaining extension problem to a simpler task, one shows [12] that it is possible to expand the $\tau^0$ in terms of the mass parameter $m$ in a "scaling expansion" of the form

$$\tau^{(m)\,0} = \sum_{k=n}^{j} m^{2k} \cdot u_k^0 + r_j^0, \tag{37}$$

where the $u_j^0$ are Poincare invariant distributions (independent of $m$) that scale almost homogeneously under a rescaling of the spacetime coordinates,

$$\left( \frac{\partial}{\partial \log \lambda} \right)^k \left\{ \lambda^{D-2k} u_k^0(\lambda x_1, \dots, \lambda x_n) \right\} = 0, \tag{38}$$

with $\mathrm{WF}(u_k^0) \subset \mathcal{C}_T$, and where the remainder $r_j^0$ is a distribution with $\mathrm{WF}(r_j^0) \subset \mathcal{C}_T$, smooth in $m$, whose scaling degree [3] can be made arbitrarily low by carrying out the

[13] We are also suppressing tensor indices in eqs. (33) and (34). For example, the notation $(\partial)^j \phi$ is a shorthand for $\partial_{(\mu_1} \dots \partial_{\mu_j)} \phi$

[14] The unitarity condition on the $T$ also implies a certain reality condition on the $\tau$, which however is rather easy to satisfy in the present context.

expansion to sufficiently large order $j$. The idea is now to construct the desired extension $\tau$ by constructing separately suitable extensions of $u_k^0$ and $r_j^0$. Actually, since the remainder has a sufficiently low scaling degree, it extends by continuity to a unique distribution $r_j$, and that extension is seen to be automatically Poincare invariant, have wave front set $\mathrm{WF}(r_j) \subset \mathcal{C}_T$, and have an almost homogeneous scaling behavior under a rescaling of the coordinates and the mass parameter. The distributions $u_k^0$, on the other hand, do not extend by continuity, but one can construct the desired extension as follows [12]: One first constructs, by the methods originally due to Epstein and Glaser and described e.g. in [19], an arbitrary extension that is translationally invariant and has the same scaling degree as the unextended distribution. That extension then also satisfies the wave front set condition [3], but it will not, in general, yield a distribution with an almost homogeneous scaling behavior (i.e., homogeneous scaling up only to logarithmic terms), nor will it be Lorentz invariant. The point is, however, that this preliminary extension can always be modified, if necessary, so as to restore the almost homogeneous scaling behavior and Lorentz invariance (while at the same time keeping the wave front set property and translational invariance), see lemma 4.1 of ref. [12]. This accomplishes the desired extension of the $\tau^0$, and thereby establishes the existence of a prescription $T$ for time ordered products satisfying (t1)–(t8).

We emphasize that the above list of properties (t1)–(t8) does not determine the Wick powers and time ordered products uniquely (for the time ordered products, this non-uniqueness arises because the extension process is not unique). The non-uniqueness corresponds to the usual "finite renormalization ambiguities". Their form is severly restricted by the properties (t1)–(t8) and is described by the "renormalization group", see section 5.

# 3 Algebraic construction of the field observables as polynomials in $1/N$

In this section, we generalize the algebraic construction of the field observables from a single scalar field to a multiplet of scalar fields, and we will show that the number of field components can be viewed as a free parameter that can be taken to infinity in a meaningful way on the algebraic level. The model that we want to consider is described by the classical action

$$S = \int \mathrm{Tr}\,(\partial^\mu \phi \partial_\mu \phi + m^2 \phi^2)\, d^d x, \tag{39}$$

where $\phi = \{\phi_{ij'}\}$ is now a field taking values in the hermitian $N \times N$ matrices, and where "Tr" denotes the trace, with no implicit normalization factors. More precisely, we should think of the field as taking values in the $\mathbf{N} \otimes \bar{\mathbf{N}}$ representation[15] of the group $U(N)$, the

[15] We are putting a prime on the indices associated with the tensor factor transforming under $\bar{\mathbf{N}}$ in the spirit of van der Waerden's notation.

trace being given by $\operatorname{Tr}\phi^a = \sum \phi_{ij'}\delta^{j'k}\phi_{kl'}\delta^{l'm}\dots\phi_{mn'}\delta^{n'i}$ in terms of the invariant tensor $\delta_{ij'}$ in $\mathbf{N}\otimes\bar{\mathbf{N}}$.

For an arbitrary, but fixed $N$, we begin by constructing a minimal algebra algebra of observables corresponding to the action (39) in a similar way as described in the previous section for the case of a single field. The minimal algebra is now generated by a unit and finite sums of products of smeared field components, $\phi_{ij'}(f)$, where $f$ runs through all compactly supported testfunctions, and where the "color indices" $i$ and $j'$ run from 1 to $N$. The relations in the case of general $N$ differ from relations 1)–4) for a single field only in that the hermiticity and commutation relations now read

$3_N$. (hermiticity) $\phi_{ij'}(f)^* = \phi_{j'i}(\bar{f})$

$4_N$. (commutator) $[\phi_{ij'}(f), \phi_{kl'}(h)] = i\delta_{il'}\delta_{kj'}\Delta(f,h)\cdot\mathbf{1}$,

where $\Delta$ is the advanced minus retarded propagator of a single Klein-Gordon field. We construct an enarged algebra, $\mathcal{W}_N$, by passing to a new set of generators of the form (3) and by allowing these generators to be smeared with suitable distributions, i.e., $\mathcal{W}_N$ is spanned by expressions of the form

$$A = \int :\prod_{k}^{a}\phi_{i_k j_k{}'}(x_k): t(x_1,\dots,x_a)\prod_{k=1}^{a} d^d x_k, \tag{40}$$

where $t \in \mathcal{E}'_a$ and where we are using the usual informal integral notation for distributions. The product of these quantities can again be expressed in a form that is similar to (5). Since the real components of the field $\phi_{ij'}$ are not coupled to each other, the enlarged algebra $\mathcal{W}_N$ is isomorphic to the tensor product of the corresponding algebra $\mathcal{W}_1$ for each independent real component of the field as defined in the previous section,

$$\mathcal{W}_N \cong \bigotimes^{N^2} \mathcal{W}_1, \tag{41}$$

$N^2$ being the number of independent real components of the field $\phi_{ij'}$.

The transformation $\phi_{ij'} \to U_i{}^k\bar{U}_{j'}{}^{l'}\phi_{kl'}$ leaves the classical action functional (39) invariant for any unitary matrix $U \in U(N)$. This invariance property is expressed on the algebraic level by a corresponding action of the group $U(N)$ on the algebras $\mathcal{W}_N$ via a group of *-automorphisms $\alpha_U$. We are interested in the subalgebra

$$\mathcal{W}_N^{\rm inv} = \{A \in \mathcal{W}_N \mid \alpha_U(A) = A \quad \forall U \in U(N)\} \tag{42}$$

of "gauge invariant" elements, i.e., the subalgebra of $\mathcal{W}_N$ consisting of those elements that are invariant under this automorphic action of the group $U(N)$. It is not difficult to convince oneself that, as a vector space, $\mathcal{W}_N^{\rm inv}$ is given by

$$\mathcal{W}_N^{\rm inv} = \operatorname{span}\{W_a(t) \mid t \in \mathcal{E}'_{|a|}\}, \tag{43}$$

where

$$W_a(t) = \frac{1}{N^{|a|/2}} \int :\mathrm{Tr}\left(\prod_{i_1 \in I_1} \phi(x_{i_1})\right) \cdots \mathrm{Tr}\left(\prod_{i_T \in I_T} \phi(x_{i_T})\right): t(x_1, \dots, x_{|a|}) \prod_{i=1}^{|a|} d^d x_i. \tag{44}$$

Here, $t \in \mathcal{E}'_{|a|}$, the symbol $a$ now stands for a multi index, $a = (a_1, \dots, a_T)$, and we are using the usual multi index notation

$$|a| = \sum_i^T a_i. \tag{45}$$

The $I_j$ are mutually disjoint index sets containing each $a_j$ elements such that $\cup_j I_j = \{1, \dots, |a|\}$. For later convenience, we have also incorporated an overall normalization factor into our definition of the generators $W_a(t)$. The generators are symmetric under exchange of the arguments within each trace and under exchange of the traces[16]. Furthermore, they satisfy a wave equation and hermiticity condition completely analogous to the ones in the scalar case,

$$W_a(\bar{t}) = W_a(t)^*, \quad W_a([1 \otimes \cdots (\partial^\mu \partial_\mu - m^2) \otimes \cdots 1]t) = 0, \tag{46}$$

where the Klein-Gordon operator acts on any of the arguments of $t$.

Since $\mathcal{W}_N^{\mathrm{inv}}$ is a subalgebra of $\mathcal{W}_N$ (i.e., closed under multiplication), the product of two such generators can again be expressed as a linear combination of these generators. In order to determine the precise form of this linear combination, one has to take care of the color indices and the structure of the traces. Since the traces in the generators (44) imply that there are "closed loops" of contractions of the color indices, there will also appear similar closed loops of index contractions in the formula for the product of two generators. Such closed index loops will give rise to combinatorical factors involving $N$. We are ultimately interested in taking the limit when $N$ goes to infinity, so we must study the precise form of this $N$-dependence.

Since the $N$-dependence arises solely from the index structure and not from the spacetime dependence of the propagators, it is sufficient for this purpose to study the "matrix model" given by the zero-dimensional version of the action functional (39),

$$S_{\mathrm{matrix}} = m^2 \mathrm{Tr}\, M^2, \tag{47}$$

where we have put $M = \phi$ in this case in order to emphasize the fact that we are dealing now with hermitian $N \times N$ matrices $M = \{M_{ij'}\}$ with no dependence upon the spacetime point (this action does not, of course, describe a quantum field theory). By analogy with eq. (3), we define the "normal ordered product" of $k$ matrix entries to be the function

$$:M_{i_1 j_1'} \cdots M_{i_k j_k'}: = (-i)^k \frac{\partial^k}{\partial J^{i_1 j_1'} \cdots \partial J^{i_k j_k'}} \mathrm{e}^{iJ\cdot M + \frac{1}{m^2} J^2}\Big|_{J=0} \tag{48}$$

[16]Another way of saying this is that the $W_a$ really act on distributions $t$ with these symmetry properties.

of the matrix entries where we have put $M \cdot J = \sum_{ij} M_{ij'} J^{ij'}$. For the first values of $k$, the definition yields $: M_{ij'} := M_{ij'}$, and $: M_{ij'} M_{kl'} := M_{ij'} M_{kl'} - \delta_{il'} \delta_{kj'} / m^2$, etc. The (commutative) product of normal ordered polynomials $: Q(\{M_{ij'}\}) :$ can be expressed in terms of normal ordered polynomials via the following version of Wick's theorem:

$$: Q_1 : \cdots : Q_r : = : \mathrm{e}^{iM \cdot \partial / \partial J} : \langle : \mathrm{e}^{-iJ \cdot \partial / \partial M} Q_1 : \cdots : \mathrm{e}^{-iJ \cdot \partial / \partial M} Q_r : \rangle_{\text{matrix}} \Big|_{J=0}, \tag{49}$$

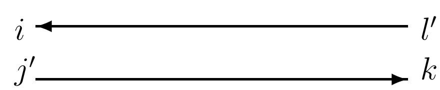
where we have introduced the "correlation functions"

$$\langle : Q_1 : \cdots : Q_r : \rangle_{\text{matrix}} \equiv \mathcal{N} \int dM \ : Q_1 : \cdots : Q_r : \ e^{-S_{\text{matrix}}} \tag{50}$$

which we normalize so that $\langle 1 \rangle_{\text{matrix}} = 1$. It follows from these definitions that we always have $\langle : Q : \rangle_{\text{matrix}} = 0$. The correlation functions can be written as a sum of contributions associated with Feynman diagrams. Each such diagram consists of $r$ vertices that are connected by "propagators"

$$\langle M_{ij'} M_{kl'} \rangle_{\text{matrix}} = \frac{1}{m^2} \delta_{il'} \delta_{kj'}, \tag{51}$$

which are represented by oriented double lines, the arrow always going from the primed to the unprimed index.




The structure of $i$-th vertix is determined by the form of the polynomial $Q_i$.

The space of gauge invariant polynomials in $M_{ij'}$ is spanned by functionals of the form[17]

$$W_a = \frac{1}{N^{|a|/2}} \ : \mathrm{Tr}\, M^{a_1} \cdots \mathrm{Tr}\, M^{a_T} : . \tag{52}$$

which is the 0-dimensional analogue of expression (44), with the only difference that there is no dependence on the smearing distribution $t$ since we are in 0 spacetime dimensions. Since these multi trace observables span the space of all polynomial $U(N)$-invariant function of the matrix entries, we already know that the product of $W_a$ with $W_b$ can again

[17]Note that these polynomials are not linearly independent at finite $N$. For example, for $N = 2$ there holds the relation $\mathrm{Tr}\, M^3 - \frac{3}{2} \mathrm{Tr}\, M \mathrm{Tr}\, M^2 + \frac{1}{2} (\mathrm{Tr}\, M)^3 = 0$. A set of linearly independent polynomials can be obtained using so-called "Schur-polynomials".

be written as a linear combination of such observables. We are interested in the dependence of the coefficients in this linear combination on $N$. We calculate the product $W_a \cdot W_b$ via Wick's formula (49), and we organize the resulting sum of expressions in terms of the following Feynman graphs. From the $T$ traces in $W_a$, there will be $T$ $a$-vertices with $a_j$ legs each (we think of the legs as carring a number), where $j = 1, \dots, T$. We draw the legs as double lines. As an example, let $a = (a_1, a_2) = (3, 3)$, so that $W_{(3,3)} = 1/N^3 : \mathrm{Tr}\, M^3 \mathrm{Tr}\, M^3 :$. In this case, we have 2 $a$-vertices with with 3 lines, each corresponding to one trace with 3 factors of $M$. Each such vertex looks therefore as follows:

The double lines should also be equipped with orientations that are compatible at the vertex, although we have not drawn this here. From the $S$ traces in $W_b$ there are similarly $S$ $b$-vertices with $b_j$ legs each, where $j = 1, \dots, S$. We consider graphs obtained by joining $a$-vertices with $b$-vertices by a double line representing a matrix propagator (51), but we do not allow any $a - a$ or $b - b$ connections (such connections have already been taken care of by the normal ordering prescription used in the definition of $W_a$ respectively $W_b$). We finally attach an "external current" $M_{ij'}$ to every leg of an $a$-vertex or $b$-vertex that is not connected by a propagator. An example of a Feynman graph resulting from this procedure occurring in the product $W_{(a_1,a_2)} \cdot W_{(b_1,b_2)}$ with $a_1 = a_2 = b_1 = b_2 = 3$ is drawn in the following picture.

The resulting structure will consist of a number of closed loops obtained by following the lines (including loops that run through external currents). There will be, in general, 3

kinds of loops: (1) Degenerate loops around a single vertex that has only external currents but no propagators attached to it. Let the number of such loops (i.e., isolated vertices) be $D$. (2) Loops that contain at least one external current and at least one propagator line. Let the number of these surfaces be $J$. (3) Loops that contain no external current. Let $I$ be the number of these loops. [Thus, in the above example graph, we have $D = 0$, $I = 1$ (corresponding to the inner square-shaped loop) and $J = 1$ (the loop running around the square passing through the 4 external currents).]

Following a set of ideas by 't Hooft [15], we consider the big (in general multiply connected) closed 2-dimensional surface $\mathscr{S}$ obtained by capping off the loops of type (2) and (3) with little surfaces (we would obtain a sphere in the above example.) The total number $F$ of little surfaces in $\mathscr{S}$ is consequently given by

$$F = I + J. \tag{53}$$

Let us label the loops containing currents by $j = 1, \dots, D + J$, and let $c_j$ be the the number of currents in the corresponding loop. By construction, the number of edges, $P$, of the surface $S$ is related to $a$, $b$, and $c$ by

$$2P = |a| + |b| - |c|, \tag{54}$$

where we are using the same multi index notation as above. The number of vertices, $V$, in $\mathscr{S}$ is

$$V = T + S - D, \tag{55}$$

i.e., is equal to the total number of traces $T$ and $S$ in $W_a$ and $W_b$ minus $D$, the number of vertices that are not connected to any other vertex. We apply the well-known theorem by Euler to the surface $\mathscr{S}$ which tells us that

$$F - P + V = \sum_k (2 - 2H_k), \tag{56}$$

where $H_k$ is the genus of the $k$-th disconnected component of $\mathscr{S}$. The little surfaces in $\mathscr{S}$ each carry an orientation induced by the direction of the enclosing index loops, and these give rise to an orientation on each of the connected components of the big surface $\mathscr{S}$. An oriented 2-dimensional surface always has $H_k \geq 0$, and $H_k$ is equal to the number of handles of the corresponding connected component in that case.

Let us analyze the contributions to the product $W_a \cdot W_b$ associated with a given graph. From the $P$ double lines of the graph, there will be a contribution

$$\prod_{\text{lines } (k,l)} \frac{1}{m^2}, \tag{57}$$

associated with the double line propagators. From the closed loops of the kind (3) in the graph there will be a factor

$$N^I = N^{\sum(2-2H_k)+(|a|+|b|-|c|)/2-V-J} \tag{58}$$

because each of the $I$ such closed index loops gives rise to a closed loop of index contractions of Kronecker deltas, $N = \sum \delta_i{}^i$. Finally, there will be a contribution

$$: \mathrm{Tr}\, M^{c_1} \cdots \mathrm{Tr}\, M^{c_{J+D}} : \tag{59}$$

corresponding to the external currents in $J + D$ closed loops of the kind (1) and (2) containing $c_i$ external currents each. Taking into account the normalization factors of $N^{-|a|/2}$ respectively $N^{-|b|/2}$ associated with $W_a$ respectively $W_b$, and letting $V_k$ be the number of vertices in the $k$-th connected component of $\mathscr{S}$, we therefore find

$$W_a \cdot W_b = \sum_{\text{graphs}} (1/N)^{J+\sum H_k + \sum (V_k - 2)} \prod_{\text{lines } (k,l)} \frac{1}{m^2} \cdot W_c, \tag{60}$$

where the sum is over all distinct Feynman graphs[18].

We can easily generalize these considerations to calculate the product of $W_a(f_1 \otimes \cdots \otimes f_{|a|})$ with $W_b(h_1 \otimes \cdots \otimes h_{|b|})$ in the case when the dimension of the spacetime is non zero. In this case, the legs of each $a$-vertex are associated with the smearing functions $f_j$ appearing in the corresponding trace in eq. (44), and the legs of every $b$-vertex are likewise associated with a smearing functions $h_k$. Every matrix propagator connecting such an $a$ and $b$ vertex then gets replaced by $\Delta_+$,

$$\frac{1}{m^2} \to \Delta_+(f_j, h_k). \tag{61}$$

Furthermore, in a given graph, the $J + D$ index loops with currents now correspond to a contribution of the form[19]

$$: \mathrm{Tr} \left( \prod^{c_1} \phi(j_i) \right) \cdots \mathrm{Tr} \left( \prod^{c_{J+D}} \phi(j_k) \right) :, \tag{62}$$

where the $j_k \in \{f_k, h_k\}$ denotes the test function associated with the corresponding external current. With these replacements, we obtain the following formula for the product of two generators $W_a(\otimes_i f_i)$ with $W_b(\otimes_i h_i)$ of the algebra $\mathcal{W}_N^{\mathrm{inv}}$:

$$\begin{aligned} &W_a(f_1 \otimes \cdots \otimes f_{|a|}) \cdot W_b(h_1 \otimes \cdots \otimes h_{|b|}) \\ &\qquad = \sum_{\text{graphs}} (1/N)^{J+\sum H_k + \sum (V_k - 2)} \prod_{\text{lines } (k,l)} \Delta_+(f_k, h_l) \\ &\qquad\qquad \cdot W_c(j_1 \otimes \cdots \otimes j_{|c|}). \end{aligned} \tag{63}$$

[18]Note that we think of the legs of $a$- and $b$-vertices as numbered, and so a graph is understood here as a graph carrying the corresponding numberings. Topologically identical graphs with distinct numberings of the legs count as different in the above sum, as well as similar sums below.

[19]To simplify, we are assuming here that $\Delta_+$ is given by the Wightman 2-point function, see eq. (10).

An entirely analogous formula is obtained if the test functions $\otimes_i f_i$ and $\otimes_j h_j$ are replaced by arbitrary distributions $t$ and $s$ in the spaces $\mathcal{E}'_{|a|}$ respectively $\mathcal{E}'_{|b|}$.

The important thing to observe about relation eq. (63) is how the coefficients in the sum on the right side depend on $N$: The numbers $H_k$ (the number of handles of the $k$-th component of the surface $\mathscr{S}$ associated with the graph) and $J$ are always non-negative. The number $V_k - 2$ is also non-negative since the number of vertices in each component, $V_k$, is by construction always greater or equal than 2. Hence, we conclude that $1/N$ appears always with a non-negative power in the coefficients on the right side of eq. (63). Since these coefficients are essentially the "structure constants" of the algebra $\mathcal{W}_N^{\mathrm{inv}}$, it is therefore possible take the large $N$ limit on the algebraic level. We now formalize this idea by constructing a new algebra which has essentially the same relations as the algebras $\mathcal{W}_N^{\mathrm{inv}}$, but which incorporates the important new point of view that $N$, or rather

$$\varepsilon = \frac{1}{N} \tag{64}$$

is not fixed, but is instead considered as a free expansion parameter that can range freely over the real numbers, including in particular $\varepsilon = 0$. We will then show that this algebra contains elements corresponding to Wick powers and their time ordered products. The construction of this new algebra therefore incorporates the $1/N$ expansion of the quantum field observables associated with the action (39), including in particular the large $N$ limit of the theory.

Consider the complex vector space $\mathcal{X}[\varepsilon]$ consisting of formal power series expressions of the form

$$\sum_{j \ge 0} \varepsilon^j W_{a_j}(t_j) \tag{65}$$

in the "dummy variable" $\varepsilon$, where the $a_j$ are multi-indices, and where the $t_j$ are taken from the space $\mathcal{E}'_{|a_j|}$ of distributions in $|a_j|$ spacetime variables defined in (7). We implement the second of relations eq. (46) by viewing the symbols $W_{a_j}(t_j)$ as depending only on the equivalence class of $t_j$ in the quotient space $\mathcal{J}_{|a_j|}$, where

$$\mathcal{J}_n = \mathcal{E}'_n / \{(1 \otimes \cdots (\partial^\mu \partial_\mu - m^2) \otimes \cdots 1)s \mid s \in \mathcal{E}'_n\}. \tag{66}$$

On the so defined complex vector space $\mathcal{X}[\varepsilon]$, we define a product by

$$\left( \sum_{j \ge 0} \varepsilon^j W_{a_j}(t_j) \right) \left( \sum_{k \ge 0} \varepsilon^k W_{b_k}(s_k) \right) = \sum_{r \ge 0} \varepsilon^r \sum_{r = k+j} W_{a_j}(t_j) \cdot W_{b_k}(s_k), \tag{67}$$

where the product $W_{a_j}(t_j) \cdot W_{b_k}(s_k)$ is given by formula eq. (63) (with $1/N$ replaced by $\varepsilon$ in this formula), and we define a *-operation on $\mathcal{X}[\varepsilon]$ by

$$\left( \sum_{j \ge 0} \varepsilon^j W_{a_j}(t_j) \right)^* = \sum_{j \ge 0} \varepsilon^j W_{a_j}(\bar{t}_j). \tag{68}$$

**Proposition 1.** The product formula (67) and the formula (68) for the *-operation makes $\mathcal{X}[\varepsilon]$ into an (associative) *-algebra with unit (given by $\mathbf{1} \equiv W_0$).

*Proof.* We need to check that the product formula (67) defines an associative product, and that the formula (68) for the *-operation is compatible with this product in the usual sense. For associativity, we consider the associator of generators

$$A(\varepsilon) = W_a(r) \cdot (W_b(s) \cdot W_c(t)) - (W_a(r) \cdot W_b(s)) \cdot W_c(t), \tag{69}$$

which we evaluate using the product formula in the order specified by the brackets. The resulting expression can be written as a finite sum of terms of the form $\sum Q_j(\varepsilon) W_{d_j}(u_j)$, where the $Q_j(\varepsilon)$ are polynomials in $\varepsilon$, and where the $u_j$ are linearly independent. But we already know $A(\varepsilon) = 0$ for $\varepsilon = 1, 1/2, 1/3$, etc., since the algebras $\mathcal{W}_N^{\text{inv}}, N = 1, 2, 3$, etc. are associative. Therefore, the $Q_j(\varepsilon)$ must vanish for these values of $\varepsilon$. Since a polynomial vanishes identically if it vanishes when evaluated on an infinite set of distinct real numbers, it follows that the $Q_j$ vanish identically, proving that $A(\varepsilon) = 0$ as a power series in $\varepsilon$. The consistency of the *-operation is proved similarly. $\square$

The construction of the algebra $\mathcal{X}[\varepsilon]$ completes our desired algebraic formulation of the $1/N$ expansion of the field theory associated with the free action (39). Our construction of $\mathcal{X}[\varepsilon]$ depends on a particular choice of the distribution $\Delta_+$, but different choices again give rise to isomorphic algebras, showing that, as an abstract algebra, $\mathcal{X}[\varepsilon]$ is independent of this choice. Indeed, let $\Delta'_+$ be another bidistribution whose antisymmetric part is $(i/2)\Delta$ which satisfies the wave equation and has wave front set $\text{WF}(\Delta'_+)$ of Hadamard form, and let $\mathcal{X}'[\varepsilon]$ be the corresponding algebra constructed from $\Delta'_+$ with generators $W'_a(t)$. Then the desired *-isomorphism from $\mathcal{X}[\varepsilon] \to \mathcal{X}'[\varepsilon]$ is given by

$$W_a(t) \to \sum_{\text{graphs}} \varepsilon^{J + \sum H_k + \sum (V_k - 2)} \cdot W'_b \left( \langle \bigotimes_{\text{lines}} F, t \rangle \right). \tag{70}$$

Here $F$ is the smooth function given by $\Delta_+ - \Delta'_+$ and a graph notation as in eq. (63) has been used: The sum is over all graphs obtained by writing down the $T$ vertices corresponding to the $T$ traces in $W_a(t), a = (a_1, \dots, a_T)$, by contracting some legs with "propagators", and by attaching "external currents" to others. If $x_i$ is the point associated with the $i$-th leg in a graph with $n$ propagator lines $(2n \le |a|)$, then

$$\langle \bigotimes_{\text{lines}} F, t \rangle (x_1, \dots, x_{|a|-2n}) = \int t(x_1, \dots, x_{|a|}) \prod_{\text{lines } (i,j)} F(x_i, x_j) \prod_{\text{legs } i} d^d x_i, \tag{71}$$

where the second product is over all legs which have a propagator attached to them. The numbers $H_k$ and $V_k$ are the number of handles respectively the number of vertices $(\ge 2)$ in the $k$-th disconnected component of the surface associated with the graph. $J$ is the

number of closed index loops associated with the graph which contain currents, and each such loop with $b_i$ currents corresponds to a trace in $W'_b$, where $b = (b_1, b_2, \dots, b_J)$. We note that this implies in particular that only positive powers of $\varepsilon$ appear in eq. (70), which is necessary in order for the right side to be an element in $\mathcal{X}[\varepsilon]$.

We finally show that $\mathcal{X}[\varepsilon]$ contains observables corresponding to the suitably normalized smeared gauge invariant Wick powers and their time ordered products. To have a reasonably compact notation for these objects, let us introduce the vector space of all formal gauge invariant expressions in the field $\phi$ and its derivatives,

$$\mathcal{V}^{\rm inv} = \text{span}\left\{ \mathcal{O} = \prod_i \text{Tr}\, \left( \prod \partial_{\mu_1} \cdots \partial_{\mu_k} \phi \right) \right\}. \tag{72}$$

If $\mathcal{O}$ is a monomial in $\mathcal{V}^{\rm inv}$, we denote by $|\mathcal{O}|$ the number of free field factors $\phi$ in the formal expression for $\mathcal{O}$, for example $|\mathcal{O}| = 6$ for the field $\mathcal{O} = \text{Tr}\, \phi^4 \text{Tr}\, \phi^2$. For a fixed $N$, the gauge invariant Wick powers are viewed as linear maps

$$\mathscr{D}(\mathbb{R}^d, \mathcal{V}^{\rm inv}) \to \mathcal{W}_N^{\rm inv}, \quad f\mathcal{O} \to \mathcal{O}(f). \tag{73}$$

Likewise, the gauge invariant time ordered products are viewed as multi linear maps

$$T : \times^n \mathscr{D}(\mathbb{R}^d; \mathcal{V}^{\rm inv}) \to \mathcal{W}_N^{\rm inv}, \quad (f_1\mathcal{O}_1, \dots, f_n\mathcal{O}_n) \to T(f_1\mathcal{O}_1 \cdots f_n\mathcal{O}_n). \tag{74}$$

The Wick powers are identified with the time ordered products with a single factor. In the previous section, we demonstrated that, in the scalar case ($N = 1$), the Wick powers and time ordered products can be constructed so as to satisfy a number of properties that we labelled (t1)–(t8). It is clear that these constructions can be generalized straightforwardly also to the case of a multiplet of scalar fields in the adjoint representation of $U(N)$ (with $N$ arbitrary but fixed) and thereby yield time ordered products with properties completely analogous to the properties (t1)–(t8) stated above for the scalar case. We would now like to investigate the dependence upon $N$ of these objects and show that, if the time ordered products are normalized by suitable powers of $1/N$, these can be viewed as elements of $\mathcal{X}[\varepsilon]$, i.e., that they can be expressed as a linear combination of $W_a(t)$, with $t$ depending only on positive powers of $\varepsilon = 1/N$.

The Wick powers $\mathcal{O}(f)$ are constructed in the same way as in the scalar case, see eq. (27). The only difference is that we need to multiply the Wick powers by suitable normalization factors depending upon $\varepsilon = 1/N$ in order to get well-defined elements of $\mathcal{X}[\varepsilon]$. Taking into account the normalization factor in the definition of the generators $W_a$, eq. (44), one sees that

$$\varepsilon^{|\mathcal{O}|/2} \mathcal{O}(f) \in \mathcal{X}[\varepsilon]. \tag{75}$$

Given that the suitably normalized Wick powers eq. (75) are elements in $\mathcal{X}[\varepsilon]$, one naturally expects that their time ordered products are also elements in $\mathcal{X}[\varepsilon]$,

$$\varepsilon^{\sum |\mathcal{O}_i|/2} T \left( \prod_i f_i \mathcal{O}_i \right) \in \mathcal{X}[\varepsilon]. \tag{76}$$

Now, if the testfunctions $f_i$ are temporally ordered, i.e., if for example the support of $f_1$ is before $f_2$, the support of $f_2$ before $f_3$ etc., then the time ordered product factorizes into the ordinary algebra product $\varepsilon^{|\mathcal{O}_1|/2}\mathcal{O}_1(f_1)\varepsilon^{|\mathcal{O}_2|/2}\mathcal{O}_2(f_2)\dots$ in $\mathcal{X}[\varepsilon]$, by the causal factorization property of the time ordered products. Therefore, since the normalized Wick powers have already been demonstrated to be elements in $\mathcal{X}[\varepsilon]$, also their product is (because $\mathcal{X}[\varepsilon]$ was shown to be an algebra). Hence, one concludes by this arguments that if $F = \otimes_i f_i$ is supported away from the union of all partial diagonals $D_I$ in the product manifold $\times^n \mathbb{R}^d$, then the corresponding time ordered product satisfies (76). However, for arbitrarily supported testfunctions $f_i$, the time ordered product does not factorize, and therefore does not correspond to the usual algebra product. In other words, whether eq. (76) is satisfied or not depends on the definition of the time ordered products on the partial diagonals $D_I$ (as a function of $N$). As we have described explicitly above in the scalar case, the definition of the time ordered products on the diagonals is achieved by extending the time ordered products defined by causal factorization away from the diagonals in a suitable way. Therefore, in order that eq. (76) be satisfied, we must control the dependence upon $N$ of the constructions in the extension argument, or, said differently, we must control the way in which the time ordered products are renormalized as a function of $N$.

Acutally, we will now argue that one can construct time ordered products $T$ satisfying eq. (76) [in addition to (t1)–(t8)] from the scalar time ordered products that were constructed in the previous section, so there is no need to repeat the extension step. The arguments are purely combinatorical and very similar to the kinds of arguments used before in the construction of the algebra $\mathcal{X}[\varepsilon]$, so we will only sketch them here. Also, to keep things as simple as possible, we will consider explicitly only the case in which all the $\mathcal{O}_i$ contain only one trace and no derivatives, $\mathcal{O}_i = \operatorname{Tr}\phi^{a_i}$. In order to describe our construction of $T(\prod_i \mathcal{O}_i)$, we begin by considering the coefficient distributions $\tau[\otimes_i \phi^{b_i}]$ occurring in the Wick expansion (33) of the time ordered products $T(\prod \phi^{a_i})$ in the *scalar* theory ($N = 1$). As it is well-known, these can be decomposed into contributions from individual Feynman graphs

$$\tau[\phi^{b_1} \otimes \cdots \otimes \phi^{b_n}] = \sum_{\text{graphs } \gamma} c^\gamma \cdot \tau^\gamma[\phi^{b_1} \otimes \cdots \otimes \phi^{b_n}]. \tag{77}$$

Here, the sum is over all distinct Feynman graphs $\gamma$ (in the scalar theory) with $n$ vertices of valence $b_i$ each (and no external lines), and $c^\gamma$ is a combinatorical factor chosen so that $\tau^\gamma$ coincides with the distribution constructed by the usual Feynman rules (where the latter are well-defined as distributions, i.e., away from the diagonals). Consider now, in $\mathcal{X}[\varepsilon]$, the product $\varepsilon^{|\mathcal{O}_1|/2}\mathcal{O}_1(f_1)\varepsilon^{|\mathcal{O}_2|/2}\mathcal{O}_2(f_2)\dots$. If one evaluates this product successively using the product formula (64), then one sees that the result is organized in terms the following double-line Feynman graphs $\Gamma$: Each such graph has $n$ vertices labelled by points $x_i$ of valence $a_i$ each (drawn as in the figure at the bottom of p. 17). External lines ending on a vertex $x_k$ carrying a color index pair $(ij')$ are associated with a factor of $\phi_{ij'}(x_k)$. Given a such a graph $\Gamma$, we form the surface $\mathscr{S}_\Gamma$ by capping off the closed index loops

with little surfaces, i.e. the index loops that do not meet any of the factors $\phi_{ij'}(x_k)$. We do not cap off any of the index loops that meet one or more of the factors $\phi_{ij'}(x_k)$ along the way, and these will consequently correspond to holes in the surface. We let $J$ be the number of such holes and we let $h = 1, \dots, J$ be an index labelling the holes. We will say that $k \in h$ if the factor $\phi_{ij'}(x_k)$ is encountered when running around the index loop in the hole labelled by $h$. Finally, for a given double line graph $\Gamma$, let $\gamma$ be the single line graph obtained from $\Gamma$ by removing all the external lines, and by replacing all remaining double lines by single ones. Then, for $(x_1, \dots, x_n)$ such that $x_i \neq x_j$ for all $i, j$ — i.e., away from all partial diagonals — we can rewrite $T(\prod \operatorname{Tr} \phi^{a_i}(x_i))$ as follows:

$$\varepsilon^{\sum a_i/2} T\left(\prod_i \operatorname{Tr} \phi^{a_i}(x_i)\right) = \sum_\Gamma \varepsilon^{J + \sum H_k + \sum (V_k - 2)} \tau^\gamma(x_1, \dots, x_n) \times \\ :\prod_h \operatorname{Tr} \left\{\prod_{k \in h} \varepsilon^{1/2} \phi(x_k)\right\} :_H, \quad (78)$$

where $V_k$ is the number of vertices in the $k$-th disconnected component of $\mathscr{S}_\Gamma$, and $H_k$ the number of handles. The idea is now to *define* the time ordered product on the left side by the right side for arbitrary $(x_1, \dots, x_n)$, including configurarions on the partial diagonals. A similar definition can be given for operators $\mathcal{O}_i$ containing multiple traces or derivatives, the only difference being that the Feynman diagrams that are involved have to also incorporate the multiple traces.

The key point about our definition (78) is that we have now complete control over the dependence upon $N$ of the time ordered products of gauge invariant elements: Since the expression in the second line of the above equation is an element of $\mathcal{X}[\varepsilon]$ (after smearing), it follows that the so-defined $\varepsilon^{\sum a_i/2} T(\prod \operatorname{Tr} \phi^{a_i}(x_i))$ is an element of $\mathcal{X}[\varepsilon]$ (after smearing). Also, since the $\tau$ have been defined so that the corresponding time ordered products in the scalar theory (see eq. (33), with $\tau^0$ replaced by $\tau$ in that equation) satisfy (t1)–(t8), it follows that the time ordered products at arbitrary $N$ defined by eq. (78) also satisfy these properties. A similar argument can be given when the operators $\mathcal{O}_i$ contain multiple traces or derivatives. Thus, we have altogether shown that the algebra $\mathcal{X}[\varepsilon]$ of formal power series in $\varepsilon = 1/N$ contains the suitably normalized time ordered products of gauge invariant elements, and that these time ordered products can be defined so that they satisfy the analogs of (t1)–(t8).

# 4 The interacting field theory

In the previous sections, we constructed an algebra of observables $\mathcal{X}[\varepsilon]$ associated with the free field described by the action (39), whose elements are (finite) power series in the free parameter $\varepsilon = 1/N$. This algebra contains, among others, the gauge invariant smeared

Wick powers of the free field and their time ordered products. In the present section we will show how to construct from these building blocks the interacting field quantities as power series in $\varepsilon$ and the self-coupling constant in an interacting quantum field theory with free part (39) and gauge invariant interaction part,

$$S = \int \mathrm{Tr}\,(\partial^\mu \phi \partial_\mu \phi + m^2 \phi^2) + V(\phi)\, d^d x. \tag{79}$$

For definiteness we consider the self-interaction

$$V(\phi) = g \mathrm{Tr}\, \phi^4 \tag{80}$$

which will be treated perturbatively.

We begin by constructing the perturbation series for the interacting fields for a given but *fixed* $N$. Let $K$ be a compact region in $d$-dimensional Minkowski spacetime, and let $\theta$ be a smooth cutoff function with which is equal to 1 on $K$ and which vanishes outside a compact neighborhood of $K$. For the cutoff interation $\theta(x)V$ and a given $N$, we define interacting fields by Bogoliubov's formula

$$\mathcal{O}_{\theta V}(f) \equiv \frac{\partial}{i\partial\lambda} S(\theta V)^{-1} S(\theta V + \lambda f \mathcal{O}) \Big|_{\lambda=0}, \tag{81}$$

where the *local* $S$*-matrices* appearing in the above equation are defined in terms of the time ordered products in the free theory by

$$S\left(g \sum_j f_j \mathcal{O}_j\right) = \sum_n \frac{(ig)^n}{n!} T\left(\prod^n \sum_j f_j \mathcal{O}_j\right), \quad \mathcal{O}_j \in \mathcal{V}^{\mathrm{inv}}. \tag{82}$$

Although each term in the power series defining the local $S$-matrix is a well defined element in the algebra $\mathcal{W}_N^{\mathrm{inv}}$, the infinite sum of these terms is not, since this algebra by definition only contains finite sums of generators. We do not want to concern ourselves here with the problem of convergence of the perturbative series, so we will view the local $S$-matrix, and likewise the interacting quantum fields (81), simply as a formal power series in $g$ with coefficients in $\mathcal{W}_N^{\mathrm{inv}}$, that is, as elements of the vector space

$$\mathcal{W}_N^{\mathrm{inv}}[g] = \left\{ \sum_{n=0}^{\infty} A_n g^n \;\middle|\; A_n \in \mathcal{W}_N^{\mathrm{inv}} \quad \forall n \right\}. \tag{83}$$

We make the space $\mathcal{W}_N^{\mathrm{inv}}[g]$ into a $*$-algebra by defining the product of two formal power series to be the formal power series obtained by formally expanding out the product of the infinite sums, $(\sum_n A_n g^n) \cdot (\sum_m B_m g^m) = \sum_k \sum_{m+n=k} (A_n \cdot B_m) g^k$, and by defining the *-operation to be $(\sum_n A_n g^n)^* = \sum_n A_n^* g^n$.

We now remove the cutoff $\theta$ on the algebraic level. For this, we first note that the coefficients in the power series[20] defining the interacting field (81) with cutoff are in fact the so-called "totally retarded products",

$$\mathcal{O}_{\theta V}(f) = \mathcal{O}(f) + \sum_{n\geq 1} \frac{(ig)^n}{n!} R(\underbrace{\theta \mathrm{Tr}\,\phi^4 \dots \theta \mathrm{Tr}\,\phi^4}_{n \text{ factors}}; f\mathcal{O}), \tag{84}$$

each of which can in turn be written in terms of products of time ordered products. It can be shown that the retarded products vanish whenever the support of $\theta$ is not in the causal past of the support of $f$. This makes it possible to define the interacting fields not only for compactly supported cutoff functions $\theta$, but more generally for cutoff functions with compact support only in the time direction, i.e., we can choose $K$ to be a time slice. Thus, when $\theta$ is supported in a time slice $K$, then the right side of eq. (84) is still a well-defined element of $\mathcal{W}^{\mathrm{inv}}_N[g]$.

We next remove the restriction to interactions localized in a time slice. For this, we consider a sequence of cutoff functions $\{\theta_j\}$ which are 1 on time slices $\{K_j\}$ of increasing size, eventually covering all of Minkowksi spacetime in the limit as $j$ goes to infinity. It is tempting to try to define the interacting field without cutoff as the limit of the algebra elements obtained by replacing the cutoff function $\theta$ in eq. (81) by the members of the sequence $\{\theta_j\}$. This limit, provided it existed, would in effect correspond to defining the interacting field in such a way that it coincides with the free "in"-field in the asymtptoic past. However, it is well-known that such an "in"-field will in general fail to make sense in the massless case due to infrared divergences. Moreover, it is clear that the local quantum fields in the interior of the spacetime should at any rate make sense no matter what the infrared behavior of the theory is. As we will see, these difficulties are successfully avoided if, instead of trying to fix the interacting fields as a suitable "in"-field in the asymptotic past, we fix them in the interior of the spacetime.

We now formalize this idea following [10] (which in turn is based on ideas of [3]). For this, it is important that for any pair of cutoff functions $\theta, \theta'$ which are equal to 1 on a time slice $K$, there exists a unitary $U(\theta, \theta') \in \mathcal{W}^{\mathrm{inv}}[g]$ such that [3]

$$U(\theta, \theta') \cdot \mathcal{O}_{\theta V}(f) \cdot U(\theta, \theta')^{-1} = \mathcal{O}_{\theta' V}(f) \tag{85}$$

for all testfunctions $f$ supported in $K$, and for all $\mathcal{O}$. These unitaries are in fact given by

$$U(\theta, \theta') = S(\theta V)^{-1} S(h_- V), \tag{86}$$

where $h_-$ is equal to $\theta - \theta'$ in the causal past of $K$ and equal to 0 in the causal future of $K$. Equation (85) shows in particular that, within $K$, the algebraic relations between the interacting fields do not depend on one's choice of the cutoff function. From our sequence

[20] This series is sometimes referred to as "Haag's series", since it was first obtained in [8].

of cutoff functions $\{\theta_j\}$, we now define $u_1 = \mathbf{1}$ and unitaries $u_j = U(\theta_j, \theta_{j-1})$ for $j > 1$, and we set $U_j = u_1 \cdot u_2 \cdot \dots \cdot u_j$. We define the interacting field without cutoff to be

$$\mathcal{O}_V(f) \equiv \lim_{j \to \infty} U_j \cdot \mathcal{O}_{\theta_j V}(f) \cdot U_j^{-1}, \tag{87}$$

where $f$ is allowed to be an arbitrary testfunction of compact support. In fact, using eqs. (85) and (86), one can show (see prop. 3.1 of [10]) that the sequence on the right side remains constant once $j$ is so large that $K_j$ contains the support of $f$, which implies that the right side is always a well-defined element of $\mathcal{W}_N^{\rm inv}[g]$. The unitaries $U_j$ in eq. (87) implement the idea to "keep the interacting field fixed in the interior of the slice $K_1$", instead of keeping it fixed in the asymptotic past. This completes our construction of the interacting fields without cutoff.

These constructions can be generalized to define time ordered products of interacting fields by first considering the corresponding quantities associated with the cutoff interaction $\theta(x)V$,

$$T_{\theta V}(f_1\mathcal{O}_1 \cdots f_n\mathcal{O}_n) \equiv \frac{\partial}{i^n \partial\lambda_1 \cdots \partial\lambda_n} S(\theta V)^{-1} S(\theta V + \sum_i \lambda_i f_i \mathcal{O}_i)\Big|_{\lambda_i = 0}, \tag{88}$$

possessing a similar expansion in terms of retarded products,

$$T_{\theta V}(\prod_i f_i \mathcal{O}_i) = T(\prod_i f_i \mathcal{O}_i) + \sum_{n \geq 1} \frac{(ig)^n}{n!} R(\underbrace{\theta \mathrm{Tr}\, \phi^4 \dots \theta \mathrm{Tr}\, \phi^4}_{n \text{ factors}}; \prod_i f_i \mathcal{O}_i). \tag{89}$$

The corresponding time ordered products without cutoff, denoted $T_V(\prod f_i \mathcal{O}_i)$, are then defined in the same way as the interacting Wick powers, see eq. (87). The latter are, of course, equal to the time ordered products with only one factor,

$$\mathcal{O}_V(f) = T_V(f\mathcal{O}). \tag{90}$$

The definition of the interacting fields as elements of $\mathcal{W}_N^{\rm inv}[g]$ depends on the chosen sequence of time slices $\{K_j\}$ and corresponding cutoff functions $\{\theta_j\}$. However, one can show (see p. 138 of [10]) that the *-algebra generated by the interacting fields does not depend these choices in the sense that different choices give rise to isomorphic algebras[21]. Furthermore, the local fields and their time ordered products constructed from different choices of $\{K_j\}$ and $\{\theta_j\}$ are mapped into each other under this isomorphism. In this sense, our algebraic construction of the interacting field theory is independent of these choices. Although this is not required in this paper, we remark that the above algebras of interacting fields can also be equipped with an action of the Poincare group on $d$-dimensional Minkowski spacetime by a group of automorphisms transforming the fields

[21] These algebras do not, of course, define the same subalgebra of $\mathcal{W}_N^{\rm inv}[g]$.

in the usual way. Thus, we have achieved our algebraic formulation of the interacting quantum field theory given by the action (79) for an arbitrary, but fixed $N$.

We will now take the large $N$ limit of the interacting field theory on the algebraic level in a similar way as in the free theory described in the previous section, by showing that the (suitably normalized) interacting fields can be viewed as formal power series in the free parameter $\varepsilon = 1/N$, provided that the 't Hooft coupling

$$g_{\mathrm{t}} = gN \tag{91}$$

is held fixed at the same time. In fact, the suitably normalized interacting fields will be shown to be elements of a subalgebra of the algebra $\mathcal{X}[\varepsilon, g_{\mathrm{t}}]$ of formal power series in $g_{\mathrm{t}}$ with coefficients in $\mathcal{X}[\varepsilon]$.

To begin, we prove a lemma about the dependence upon $N$ of the $n$-th order contribution to the the interacting field with cutoff interaction, given by the $n$-th retarded product in eq. (89).

**Lemma 1.** Let $\varepsilon = 1/N$, $\mathcal{O}_i, \Psi_j \in \mathcal{V}^{\mathrm{inv}}$, let $f_i, h_j \in \mathscr{D}(\mathbb{R}^d)$, and let $T_i$ respectively $S_j$ be the number of traces occurring in $\mathcal{O}_i$ respectively $\Psi_j$. Then we have

$$R(\prod_{i=1}^{n} \varepsilon^{-2+T_i+|\mathcal{O}_i|/2} f_i \mathcal{O}_i ; \prod_{j=1}^{m} \varepsilon^{S_j+|\Psi_j|/2} h_j \Psi_j) = O(1), \tag{92}$$

where the notation $O(\varepsilon^k)$ means that the corresponding algebra element of $\mathcal{W}_N^{\mathrm{inv}}$ (supposed to be given for all $N$) can be written as a linear combination of the generators $W_a(t)$ with $t$ independent of $\varepsilon$, and with coefficients of order $\varepsilon^k$.

*Proof.* For simplicity, we first give a proof of eq. (92) in the case $m = 1$; the case of general $m$ is treated below. Let us define, following [4], the "connected product" in $\mathcal{W}_N^{\mathrm{inv}}$ as the $k$-times multilinear maps on $\mathcal{W}_N^{\mathrm{inv}}$ defined recursively by the relation

$$(W_{a_1}(t_1) \cdot \dots \cdot W_{a_k}(t_k))^{\mathrm{conn}} \equiv W_{a_1}(t_1) \cdot \dots \cdot W_{a_k}(t_k) - \sum_{\{1,\dots,k\} = \cup I} \prod_{I}^{\mathrm{class}} \left( \prod_{j \in I} W_{a_j}(t_j) \right)^{\mathrm{conn}}, \tag{93}$$

where the "classical product" $\cdot_{\mathrm{class}}$ is the commutative associative product on $\mathcal{W}_N^{\mathrm{inv}}$ defined by

$$W_a(t) \cdot_{\mathrm{class}} W_b(s) = W_{ab}(t \otimes s), \tag{94}$$

and where the trivial partition $I = \{1, \dots, k\}$ is excluded in the sum. We now analyze the $\varepsilon$-dependence of the contracted product, restricting attention for simplicity first to the case when each of the $W_{a_i}(t_i)$ contains only one trace. We use the product formula (63) to evaluate the connected product $(W_{a_1}(t_1) \cdot \dots \cdot W_{a_k}(t_k))^{\mathrm{conn}}$ as a sum of contributions of the form $\varepsilon^I W_b(s)$ associated with Feynman graphs $\Gamma$, where $I$ is the number of index loops

in the graph, and where $s \in \mathcal{E}'_{|b|}$ does not depend upon $\varepsilon$. It is seen, as a consequence of our definiton of the connected product, that precisely the connected diagrams occur in the sum. By arguments similar to the one given in the previous section, the number $I$ associated with a given conneceted diagram with $k$ vertices is given by $2 - k - H - J$, where $H$ is the number of handles of the surface associated with the diagram, and where $J$ is the number of traces in the algebraic element $W_b(s)$ associated with the contribution of that Feynman graph. Consequently, since $H, J \geq 0$, we have

$$(W_{a_1}(t_1) \cdot \dots \cdot W_{a_k}(t_k))^{\text{conn}} = O(\varepsilon^{k-2}) \tag{95}$$

when each of the $W_{a_i}(t_i)$ contains only one trace. Now consider the retarded product when each of the fields has only one trace, and when the supports of the testfunctions $f_i, h$ satisfy

$$\text{supp} f_i \cap \text{supp} h = \text{supp} f_i \cap \text{supp} f_j = \emptyset. \tag{96}$$

Without loss of generality, we can assume that the supports of the $f_i$ have no intersection with either the causal past or the causal future of the support of $h$ (otherwise, we write each $f_i$ as a sum of two testfunctions with this property). Under these assumptions, the retarded product is given by [4]

$$R(\prod_{i=1}^{n} f_i \mathcal{O}_i; h\Psi) = \sum_{\pi} [\mathcal{O}_{\pi 1}(f_{\pi 1}), [\mathcal{O}_{\pi 2}(f_{\pi 2}), \dots [\mathcal{O}_{\pi n}(f_{\pi n}), \Psi(h)] \dots ]], \tag{97}$$

when the supports of all $f_i$ have no point in common with the causal future of the support of $h$, and by 0 otherwise. We now use the following lemma which we are going to prove below:

**Lemma 2.** Let $B, A_1, \dots, A_n \in \mathcal{W}_N^{\text{inv}}$. Then

$$\sum_{\pi} \left( [A_{\pi n}, [A_{\pi(n-1)}, \dots [A_{\pi 1}, B] \dots ]] \right)^{\text{conn}} = \sum_{\pi} [A_{\pi n}, [A_{\pi(n-1)}, \dots [A_{\pi 1}, B] \dots ]]. \tag{98}$$

Since $\varepsilon^{|\mathcal{O}_i|/2} \mathcal{O}_i(f_i)$ and $\varepsilon^{|\Psi|/2} \Psi(h)$ can be written in the form $W_a(t)$ for some distributions $t$ not depending on $\varepsilon$, it follows by eqs. (95) and (97) and the lemma that

$$R(\prod_{i=1}^{n} \varepsilon^{|\mathcal{O}_i|/2} f_i \mathcal{O}_i, \varepsilon^{|\Psi|/2} h\Psi) = O(\varepsilon^{n-1}) \tag{99}$$

when the supports of $f_i, h$ satisfy eq. (96), and when each of the fields $\mathcal{O}_i, \Psi$ contains only one trace. When the testfunctions $f_i, h$ have overlapping supports, the formula (97) for the retarded products is not well-defined, or, alternatively speaking, the formula only defines an algebra valued distribution on the domain

$$\times^{n+1} \mathbb{R}^d \setminus \bigcup_{I \subset \{1, \dots, n+1\}} D_I, \tag{100}$$

where $D_I$ is a "partial diagonal" in the product manifold $\times^{n+1}\mathbb{R}^d$, see eq. (32). However, as explained at the end of section 3, we are considering a prescription for constructing the time ordered (and hence retarded) possessing a Wick expansion of the form eq. (78) everywhere, including the diagonals. The Wick expansion eq. (78) implies that the $N$-dependence on the diagonals is identical to that off the diagonals. Equation (99) therefore follows immediately for all test functions. This proves the desired relation (92) when $m = 1$ and when all fields contain only one trace.

The situation is a bit more complicated when the fields $\mathcal{O}_i, \Psi$ contain multiple traces. In that case, we similarly begin by analyzing the $N$-dependence of the connected product (93) when the $W_{a_i}(t_i)$ contain multiple traces, so that each $a_i$ now stands for a multi index $(a_{i1}, \dots, a_{iT_i})$, where $T_i$ is the number of traces in $W_{a_i}(t_i)$, and where $a_{ij}$ is the number of free field factors appearing in the $j$-th trace of $W_{a_i}(t_i)$. It is seen that only the following type of Feynman graphs can occur in the connected product of these algebra elements: The valence of the vertices of the graphs are determined by the number of fields $a_{ij}$ appearing in the $j$-th trace of the $i$-th algebra element. For each fixed $i$, no $a_{ij}$-vertex can be connected to a $a_{ik}$-vertex. For fixed $i, l$, there exist indices $j, k$ such the $a_{ij}$-vertex is connected to the $a_{lk}$-vertex. Analyzing the $N$-dependence of these graphs arising from index contractions along closed index loops in same way as in our analysis of the $N$-dependence of the algebra product (63), we find that the contributions from these Feynman graphs are at most of order

$$O(\varepsilon^{J+2\sum H_j+\sum V_j-2C}), \tag{101}$$

where $C$ is the number of disconnected components of the surface associated with the graph, $J$ is the number of closed index loops containing "external currents", $V_j$ is the number of vertices in the $j$-th disconnected component, and $H_j$ the number of handles (components containing only a single vertex do not count). Clearly, we have $H_j, J \geq 0$ and we know that

$$\sum V_j = \sum T_i - D, \tag{102}$$

with $D$ the number of vertices that are not connected to any other vertex. In order to estimate the number $C$ of connected components of the graph, we first assume $D = 0$ and imagine the graph obtained by moving all the $a_{ij}, j = 1, \dots, T_i$ on top of each other for each $i$. The resulting structure will then only have one connected component, since we know that for fixed $i, l$, there exist indices $j, k$ such the $a_{ij}$-vertex is connected to the $a_{lk}$-vertex. If we now move the $a_{ij}, j = 1, \dots, T_i$ apart again for a given $i$, then it is clear that we will create at most $T_i - 1$ new disconnected components. Doing this for all $i$, we therefore see that our graph can have at most $1 + (T_1 - 1) + \cdots + (T_k - 1)$ disconnected components. If $D$ is not zero, then we repeat this argument for those vertices that are not isolated, and we similarly arrive at the estimate

$$C \leq 1 - k + \sum T_i - D. \tag{103}$$

for the number of disconnected components of any graph appearing in the connected product (93). Hence, we find altoghether that

$$(W_{a_1}(t_1)\cdot\dots\cdot W_{a_k}(t_k))^{\rm conn} = O(\varepsilon^{2k-2-\sum T_i}) \tag{104}$$

when each of the $W_{a_j}(t_j)$ contains $T_j$ traces. We can now finish the proof in just the same way as in the case when all the fields $\mathcal{O}_i, \Psi$ contain only a single trace.

Now let $m$ in eq. (92) be arbitrary and consider a situation wherein the supports of the testfunctions $f_i, h_j$ satisfy

$$\text{supp} f_i \cap \text{supp} h_j = \text{supp} f_i \cap \text{supp} f_j = \text{supp} h_i \cap \text{supp} h_j = \emptyset. \tag{105}$$

Without loss of generality, we assume that the support of $f_{i+1}$ has no intersection with the causal future of the support of $f_i$. Then it follows from the recursion formula (74) of [4] together with the causal factorization property of the time ordered products (17) that

$$\begin{aligned} R(\prod_{i=1}^n f_i\mathcal{O}_i; \prod_{j=1}^m h_j\Psi_j) &= \sum_\pi [\mathcal{O}_{\pi 1}(f_{\pi 1}), [\mathcal{O}_{\pi 2}(f_{\pi 2}), \dots [\mathcal{O}_{\pi n}(f_{\pi n}), \Psi_1(h_1)\cdots\Psi_m(h_m)]\dots]] \\ &= \sum_{I_1\cup\dots\cup I_m=\{1,\dots,n\}} \prod_{k=1}^m \left(\prod_{i\in I_k} \text{ad}(\mathcal{O}_i(f_i))\right)[\Psi_k(h_k)] \end{aligned} \tag{106}$$

when the supports of all $f_i$ have no point in common with the causal future of the supports of $h_j$, and by 0 otherwise, and where we have set $\text{ad}(A)[B] = [A, B]$. Since $N^{-|\mathcal{O}_i|/2}\mathcal{O}_i(f_i)$ and $N^{-|\Psi_j|/2}\Psi_j(h_i)$ can be written in the form $W_a(t)$ for some distribution not depending on $N$, we conclude by the same arguments as above that

$$\left(\prod_{i\in I_k}\text{ad}(\mathcal{O}_i(f_i))\right)[\Psi_k(h_k)] = O(\varepsilon^{2|I_k|-\sum_{i\in I_k}(T_i+|\mathcal{O}_i|/2)-(S_k+|\Psi_k|/2)}), \tag{107}$$

from which the statement of the theorem follows when the supports of $f_i, h_j$ have the properties (105). The general case can be proved from this as above.

We end the proof of lemma 1 with the demonstration of lemma 2: Let $A = \sum\lambda_i A_i$ and consider the formal power series expression

$$e^{-A}\cdot B\cdot e^A = \sum_{m,n\ge 0}\frac{1}{m!n!}(-A)^m\cdot B\cdot A^n. \tag{108}$$

For a fixed $k > 0$, consider the contribution to the sum on the right hand side arising from diagrams such that precisely $k$ $A$-vertices are disconnected from the other $A$- and $B$-vertices. Since disconnected diagrams factorize with respect to the classical product $\cdot_{\rm class}$ this contribution is seen to be equal to

$$\sum_{m,n\ge 0}\frac{1}{m!n!}\sum_{r+s=k}\frac{m!n!}{r!(m-r)!s!(n-s)!}((-A)^r\cdot A^s)\cdot_{\rm class}((-A)^{m-r}\cdot B\cdot A^{n-s}). \tag{109}$$

But this expression vanishes, due to $\sum_{r+s=k}(-A)^r \cdot A^s/r!s! = 0$, showing that $e^{-A} \cdot B \cdot e^A = (e^{-A} \cdot B \cdot e^A)^{\rm conn}$. The statement of the lemma is obtained by differentiating this expression $n$ times with respect to the parameters $\lambda_i$. $\square$

Applying the lemma to the retarded products appearing in the definition (84) of the interacting field with cutoff, (i.e., $\mathcal{O}_i = {\rm Tr}\, \phi^4$, so that $T_i = 1, |\mathcal{O}_i| = 4$ in that case), and using our assumption $g \propto \varepsilon$ (see eq. (91)), we get

$$g^n R(\prod^n \theta {\rm Tr}\, \phi^4; f\mathcal{O}) = O(\varepsilon^{-T-|\mathcal{O}|/2}), \tag{110}$$

where $T$ is the number of traces in the field $\mathcal{O}$. Therefore, since the cutoff interacting field is a sum of such terms, we have found $\mathcal{O}_{\theta V}(f) = O(\varepsilon^{-T-|\mathcal{O}|/2})$ for the cutoff interacting fields, viewed now as formal power series in the 't Hooft coupling parameter $g_{\rm t}$ rather than $g$. For the interacting time ordered products with cutoff, we similarly get $T_{\theta V}(\prod \mathcal{O}_i(f_i)) = O(\varepsilon^{-\sum T_i + |\mathcal{O}_i|/2})$. We claim that the same is true for the interacting fields without cutoff:

**Proposition 2.** Let $\varepsilon = 1/N$, $\mathcal{O} \in \mathcal{V}^{\rm inv}$ with $n$ factors of $\phi$ and $T$ traces. Then

$$\mathcal{O}_V(f) = O(\varepsilon^{-T-n/2}) \tag{111}$$

as formal power series in the 't Hooft coupling $g_{\rm t}$. More generally, for the interacting time ordered products

$$T_V(\prod \mathcal{O}_i(f_i)) = O(\varepsilon^{-\sum T_i + n_i/2}), \tag{112}$$

where $T_i$ is the number of traces in $\mathcal{O}_i$, and where $n_i$ is the number of factors of $\phi$ in $\mathcal{O}_i$.

*Proof.* According to our definition of the interacting field without cutoff, eq. (87), we must show that

$$\varepsilon^{n/2+T} \cdot U_j \cdot \mathcal{O}_{\theta_j V}(f) \cdot U_j{}^{-1} = O(1) \tag{113}$$

where $\{\theta_j\}$ and $\{U_j\}$ are sequences of cutoff functions and unitary elements as in our definition of the interacting field, see eq. (87). We expand $U_j$ and $\mathcal{O}_{\theta_j V}(f)$ in terms of the retarded products and use the fact, shown in [4], that only connected diagrams contribute to each term in the resulting formal power series. The $N$-dependence of these terms can then be analyzed in a similar fashion as in the proof of lemma 1 and gives (113).[22] The proof for the time ordered producs is similar. $\square$

The proposition allows us to view the suitably normalized interacting fields and their time ordered products as elements of the algebra $\mathcal{X}[\varepsilon, g_{\rm t}]$ of formal power series in $g_{\rm t}$ with coefficients in $\mathcal{X}[\varepsilon]$, i.e., we have shown

$$\varepsilon^{T+n/2}\mathcal{O}_V(f) \in \mathcal{X}[\varepsilon, g_{\rm t}], \tag{114}$$

[22]Note, however, that the expansion of $U_j$ itself contains *negative* powers of $\varepsilon$, i.e., it is *not* true that $U_j$ is of $O(1)$ separately.

and similarly for the interacting time ordered products[23]. We denote by $\mathcal{A}_V$ the subalgebra of $\mathcal{X}[\varepsilon, g_{\rm t}]$ generated by the fields (114) and their time ordered products,

$$\mathcal{A}_V = \text{ alg}\left\{ \varepsilon^{\sum T_i + |\mathcal{O}_i|/2} \cdot T_V(\prod_i f_i \mathcal{O}_i) \;\middle|\; f_i \in \mathscr{D}(\mathbb{R}^d), \mathcal{O}_i \in \mathcal{V}^{\rm inv} \right\} \subset \mathcal{X}[\varepsilon, g_{\rm t}]. \tag{115}$$

By the same arguments as given on p. 138 of [10], one can again prove that, as an abstract algebra, $\mathcal{A}_V$ does not depend on the choice of the cutoff functions entering in the definition of the interacting field. Since the algebra $\mathcal{A}_V$ is an algebra of formal power series in $\varepsilon = 1/N$, the construction of $\mathcal{A}_V$ accomplishes the desired algebraic formulation of the $1/N$-expansion for the interacting quantum field theory associated with the action (79).

Since the algebra $\mathcal{A}_V$ was constructed perturbatively, it incorporates not only an expansion in $1/N$, but also of course a formal expansion in the coupling parameters. Moreover, one can show that the value of Planck's constant, $\hbar$, (set equal to 1 so far) can be incorporated explicitly into the algebra $\mathcal{A}_V$, and it is seen that the classical limit, $\hbar \to 0$ can thereby included into our algebraic formulation. Following [4], we briefly describe how this is done. One first introduces an explicit dependence on $\hbar$ into the algebra product (5) in $\mathcal{W}$ by replacing $\Delta_+$ in that product formula by $\hbar\Delta_+$. With this replacement understood, $\mathcal{W}$ can now be viewed as a 1-parameter family of *-algebras depending on the parameter $\hbar$. It is possible to set $\hbar = 0$ on the algebraic level. In this limit, $\mathcal{W}$ becomes a commutative algebra, and $\frac{1}{i\hbar}$ times the commutator defines a Poisson bracket in the limit. In this way, the 1-parameter family of algebras $\mathcal{W}$ depending on $\hbar$ is seen to be a deformation of the classical Poisson algebra associated with the free Klein-Gordon field. These consideration can be generalized straightforwardly to the algebras $\mathcal{X}[\varepsilon]$ as well as $\mathcal{X}[\varepsilon, g_{\rm t}]$, and we incorporate the dependence on $\hbar$ of these algebras into the new notation $\mathcal{X}[\varepsilon, g_{\rm t}, \hbar]$. The algebras of interacting fields, $\mathcal{A}_V$, with interaction now taken to be $\frac{1}{\hbar}V$, can be seen [24] to be subalgebras of $\mathcal{X}[\varepsilon, g_{\rm t}, \hbar]$, and therefore depend likewise on the indicated deformation parameters,

$$\mathcal{A}_V = \mathcal{A}_V[\varepsilon, g_{\rm t}, \hbar]. \tag{116}$$

The interacting field algebras consequently have a classical limit, $\hbar \to 0$, and can thereby be seen to be non-commutative deformations of the Poisson algebras of classical (perturbatively defined) field observables associated with the action (79), that depend on $1/N$ as a free parameter. In this way, the expansion of the large $N$ interacting field theory in terms of $\hbar$ is incorporated on the algebraic level, and the classical limit $\hbar \to 0$ can be taken on this level. On the other hand, one can show that the vacuum state and the

[23]Note that the $\varepsilon$-dependence of the normalization factors necessary to make the interacting fields and their time ordered products elements of $\mathcal{X}[\varepsilon, g_{\rm t}]$ differs from that in the free field theory, see (75).

[24]This is a non-trivial statement, because $\frac{1}{\hbar}V$ contains *negative* powers of $\hbar$. The proof of this statement can be adapted from [4].

Hilbert space representations of $\mathcal{A}_V$ as operators on Hilbert space *cannot* be taken. This demonstrates the strength of the algebraic viewpoint.

For a more general interaction

$$V(\phi) = \sum g_i \mathcal{O}_i \tag{117}$$

including interaction vertices $\mathcal{O}_i \in \mathcal{V}^{\text{inv}}$ with $T_i$ multiple traces, it follows from lemma 1 that the interacting field will still satisfy eq. (114), provided that the coupling constants $g_i$ tend to zero for large $N$ in such a way that the corresponding 't Hooft parameters $g_{i\text{t}}$ defined by

$$g_{i\text{t}} = g_i N^{T_i + |\mathcal{O}_i|/2 - 2} \tag{118}$$

remain fixed (note that (91) is the special case $\mathcal{O}_i = \operatorname{Tr} \phi^4$ of this relation). Thus, if the coupling constants $g_i$ are tuned in the prescribed way, the interacting field algebra $\mathcal{A}_V$ is defined as a subalgebra of the algebra $\mathcal{X}[\varepsilon, g_{1\text{t}}, g_{2\text{t}}, \dots]$ of formal power series in the 't Hooft coupling parameters with coefficients in $\mathcal{X}[\varepsilon]$.

The perturbative expansion of the interacting fields (114) defined by the interaction (80) as an element of $\mathcal{A}_V$ is organized in terms of Feynman graphs that are associated with Riemannian surfaces, where contributions from genus $H$ surfaces are suppressed by a factor $\varepsilon^H$. To illustrate this in an example, consider the interacting field $\varepsilon^{3/2}(\operatorname{Tr}\phi)_{\theta V}$ with cutoff interaction $\theta(x)V$. In order to have a compact notation for the decomposition of the $n$-th order retarded product occuring in the perturbative expansion of this interacting field into contributions associated with Feynman graphs, we first consider a corresponding retarded product occurring in $\phi_{\theta V}$ in the theory of a single scalar field with interaction $\theta(x)V$, where $V = g\phi^4$. Such a retarded product can be decomposed in the form[19]

$$R(V(y_1)\cdots V(y_n); \phi(x)) = g^n \sum_{\text{graphs } \Gamma} r_\Gamma(y_1, \dots, y_n; x) : \phi^{a_1}(y_1) \cdots \phi^{a_n}(y_n) :_H . \tag{119}$$

The sum is over all connected graphs $\Gamma$ with 4-valent vertices $y_i$ and a 1-valent vertex $x$, and $a_i$ is the number of external legs (i.e., lines with open ends) attached to the vertex $y_i$. The $r_\Gamma$ are c-number distributions associated with the graph which are determined by appropriate Feynman rules.

We now look at a correpsonding retarded product occuring in the perturbative expansion of the corresponding field $\varepsilon^{3/2}(\operatorname{Tr}\phi)_{\theta V}$ in the large $N$ interacting quantum field theory with $V = g\operatorname{Tr}\phi^4$. By an analysis analogous to the one given in the proof of lemma 1, it can be shown that such a retarded product can be written as a sum of contributions from individual Feynman graphs as follows:

$$\begin{aligned}\varepsilon^{3/2} R(V(y_1)\cdots V(y_n); \operatorname{Tr}\phi(x)) = g_{\text{t}}^n \sum_{\text{genera } H} \varepsilon^H \sum_{\text{graphs } \Gamma} \varepsilon^{f/2+T} \\ \cdot r_\Gamma(y_1, \dots, y_n; x) \ : \operatorname{Tr} \prod_i \phi(y_i) \cdots \operatorname{Tr} \prod_j \phi(y_j) :_H .\end{aligned} \tag{120}$$

The expression on the right side is to be understood as follows: $g_{\rm t}$ is the 't Hooft coupling (91). The sum is over all distinct Feynman graphs $\Gamma$ that occur in the corresponding expansion (119) in the theory with only a *single* scalar field, and the c-number distributions $r_\Gamma$ are identical to the ones appearing in that expansion. The sum over graphs is subdivided into contributions grouped together according to their topology specified by the genus, $H$, of the graph, defined as the number of handles of the surface $\mathscr{S}$ obtained by attaching faces to the closed index loops occuring in the given graph (we assume that a double line notation as described in section 3 is used for the propagators and the vertices). The external legs are incorporated by capping off each such external line connected to $y_k$ and ending on the index pair $ij'$ with an "external current" $\phi_{ij'}(y_k)$. The external currents are collected in the normal ordered term appearing in eq. (120), where each trace corresponds to following through the index line to which the currents within that trace belong. The number of traces in such a normal ordered term is denoted $T$, and the number of factors of $\phi$ is denoted $f$.

The same remarks also apply to the perturbative expansion of the more general gauge invariant fields $\varepsilon^{|\mathcal{O}|/2+T}\mathcal{O}_{\theta V}$ in the large $N$ theory. A similar expansion is also valid for the corresponding fields without cutoff $\theta$. Moreover, $V$ may be replaced by an arbitrary (possibly non-renormalizable) local interaction of the form (117), provided that the couplings are tuned in the large $N$ limit in the manner prescribed in eq. (118).

# 5 Renormalization group

Our construction of the interacting field theory given in the previous section is equally valid for interactions $V$ that are renormalizable by the usual power counting criterion as well as for non-renormalizable theories. Let us sketch how the distinction between renormalizable and non-renormalizable theories appears in the algebraic framework that we are working in. For simplicity, let us first consider the theory of a single hermitian scalar field, $\phi$. We take the action of this scalar field to consist of a free part given by eq. (1), and an interaction given by $V = \sum g_i \mathcal{O}_i$, which might be renormalizable or non-renormalizable. (As above, $\mathcal{O}_i$ are monomials in $\phi$ and its derivatives.) The difference between renormalizable $V$ and non-renormalizable $V$ shows up in the perturbatively defined interacting quantum field theory as follows: Our definition of interacting fields depends on a prescription for defining the Wick powers and their time ordered products in the free theory, which is given by a map $T$ with the properties (t1)–(t8) specified in section 3. As explained there, these properties do not, in general, determine the time ordered products (i.e., the map $T$) uniquely, and this consequently leaves a corresponding ambiguity in the definition of the interacting fields. However, as first shown[25] in [10], the algebra of interacting fields

[25] The constructions in [10] were actually given in the more general context of an interacting (scalar) field theory on an arbitrary globally hyperbolic curved spacetime. An explicit treatment of the special case of Minkowski spacetime was recently given in [6].

associated with interaction $V$ constructed from a given prescription $T$ is isomorphic to the algebra constructed from any other prescription, $T'$, provided the interaction is also changed from $V$ to $V' = \sum g_i' \mathcal{O}_i$, where each of the modified couplings $g_i'$ is a suitable formal power series in the couplings $g_1, g_2, \dots$. Renormalizable theories are characterized by the fact that $V'$ always has the same form as $V$, modulo terms of the form already present in the free Lagrangian.

If $\mathcal{O}_V$ are the interacting fields constructed from the interaction $V$ using the first prescription for defining time ordered products in the free theory, and if $\mathcal{O}'_{V'}$ are the fields constructed from the interaction $V'$ and the second prescription, then the above isomorphism, let us call it $R$, can be shown [10] to be of the form

$$R : \mathcal{O}_{iV} \to \sum_j Z_{ij} \cdot \mathcal{O}'_{jV'}, \tag{121}$$

where we have omitted the smearing functions for simplicity. The "field strength renormalization" constants $Z_{ij}$ are formal power series in $g_1, g_2, \dots$. For renormalizable theories, one can show that there will appear only finitely many terms in the sum on the right side. The possible terms are restricted in that case by the requirement that the fields $\mathcal{O}_j$ on the right side cannot have a greater engineering dimension than the field $\mathcal{O}_i$ on the left side. In a non-renormalizable theory, no such restriction occurs. The map $R$ together with the transformation $V \to V'$ corresponds to the "renormalization group" in other approaches. Since the interactions $V$ might be viewed as elements of the abstract vector space $\mathcal{V}$ spanned by the field monomials $\mathcal{O}$, we may view the renormaliztion group as providing a map $\mathcal{V} \to \mathcal{V}$. The subspace of renormalizable interaction vertices $V \in \mathcal{V}$ thus correpsonds precisely to the largest finite dimensional subspace of $\mathcal{V}$ that is invariant under all renormalization group transformations.

One can in particular consider the special case in which the alternate prescription $T'$ is related to the original prescription, $T$, for defining the time ordered products in the free theory by a multiplicative change of scale (with multiplication factor $\lambda > 0$), i.e., $T'$ is given terms of $T$ by eq. (20). In that case, we obtain a family of isomorphisms $R(\lambda)$ labelled by the parameter $\lambda$, together with one-parameter families $g_i' = g_i(\lambda)$, $V' = \sum g_i(\lambda)\mathcal{O}_i$ and $Z_{ij}(\lambda)$ (for details, we refer to [10]). By the almost homogeneous scaling behavior of the time ordered products in the free theory, eq. (21), it the follows that each term appearing in the power series expansions of $g_i(\lambda)$ and $Z_{ij}(\lambda)$ depends at most *polynomially on* $\ln \lambda$. The functions $\lambda \to g_i(g_1, g_2, \dots, \lambda)$ define the "renormalization group flow" of the theory, which may be viewed as a 1-parameter family (in fact, group) of diffeomorphisms on $\mathcal{V}$. Thus, our formulation of the renormalization group flow is that a given way of defining the interacting fields $\mathcal{O}_V$ (i.e., using a given renormalization prescription) is equivalent, via the isomorphism $R(\lambda)$, to defining the fields $\mathcal{O}'_{V'}$ via the "rescaled" prescription — denoted by "prime" — obtained from the previous prescription by changing the "scale" according to eq. (20), provided that the interaction is at the same time modified to $V' = \sum g_i(\lambda)\mathcal{O}_i$.

We can re-express this renormalization group flow in a somewhat more transparent way by noting that, from eq. (20), the rescaled prescription (i.e., the "primed" prescription appearing in the renormalization group flow (121)) is given in terms of the original one (up to the isomorphism $\sigma_\lambda$) simply by appropriately rescaling the mass, the field strenght and the coordinates in the time ordered products in the free theory. Thus, by composing $R(\lambda)$ with $\sigma_\lambda$, we get the following equivalent version of our algebraic formulation of the renormalization group flow: Let $\mathcal{A}_V(U)$ be the algebra of interacting fields smeared with testfunctions supported in a region $U \subset \mathbb{R}^d$ of Minkowski space. Then $\rho_\lambda = R(\lambda) \circ \sigma_\lambda$ is given by

$$\begin{aligned} \rho_\lambda : \mathcal{A}_V^{(m)}(\lambda U) &\rightarrow \mathcal{A}_{V(\lambda)}^{(\lambda^{-1}m)}(U), \\ \mathcal{O}_{iV}(\lambda x) &\rightarrow \sum_j \lambda^{-d_j} Z_{ij}(\lambda) \cdot \mathcal{O}_{jV(\lambda)}(x) \end{aligned} \tag{122}$$

and is again an isomorphism, where we are now indicating the dependence of the algebras upon the mass parameter, $m$. Here, $V(\lambda) = \sum \lambda^{-\delta_i} g_i(\lambda) \mathcal{O}_i$, where $d_i$ is the engineering dimension of the field $\mathcal{O}_i$ and $\delta_i$ the engineering dimension of the corresponding coupling $g_i$, and the functions $Z_{ij}(\lambda), g_i(\lambda)$ are as in eq. (121). Stated differently, the action of $\rho_\lambda$ is described as follows: If the argument of an interacting field is rescaled by $\lambda$, this is equivalent via $\rho_\lambda$ to a redefinition of the interaction, $V \rightarrow V(\lambda)$ together with a suitable redefinition of the field strength by the matrix $\lambda^{-d_j} Z_{ij}(\lambda)$. We also note explicitly that eq. (122) makes reference to only *one* given renormalization prescription.

An important feature of our algebraic formulation of the renormalization group flow is that it is given directly in terms of the interacting field *operators* which are members of the algebra $\mathcal{A}_V$, rather than in terms of the correlation functions of these objects, as is normally done. Of course, one can always apply a state (i.e., a normalized linear functional on the field algebra) to the relation (121) and thereby obtain a relation for the behavior of the Green's functions under a rescaling. Our algebraic formulation makes it clear that *the existence of the renormalization group flow is an algebraic property of the theory, i.e. it is encoded in the local algebraic relations between the quantum fields. It has nothing to do a priori with the vacuum state or e.g. the superselection sector of the theory.* Besides offering a conceptually new perspective on the nature of the renormalization group flow, our algebraic formulation has the advantage that, since the construction is essentially of a local nature, it works regardless of what the infra-red behavior of the theory is. This makes the algebraic approach superior e.g. in curved spacetime [10], where there is no preferred vacuum state, and where moreover the infra-red behavior of generic states is very difficult to control (and at any rate, depends upon the behavior of the spacetime metric at large distances).

The statements just made for the theory of a single, scalar field carry over straightforwardly to a multiplet of scalar fields. In particular, they are true for the theory of a field

$\phi$ in the $\mathbf{N} \otimes \bar{\mathbf{N}}$ representation of the group $U(N)$ with action (79), for any arbitrary but fixed $N$.

The aim of the present section is to show that, for gauge invariant interactions, the algebraic formulation of renormalization group carries over in a meaningful way in the limit of large $N$, or more properly, that the renormalization group it can be defined in the sense of power series in $\varepsilon = 1/N$, with positive powers. For this, consider two different prescriptions $T$ and $T'$ for defining Wick powers and time ordered products in the free theory satisfying (t1)–(t8), as well as eq. (76). An explicit construction of such a prescription was given at the end of section 3, but we will not need to know the details of that construction here. For a given gauge invariant interaction $V = \sum g_i \mathcal{O}_i \in \mathcal{V}^{\rm inv}$ (renormalizable or non-renormalizable), let $\mathcal{A}_V$ respectively $\mathcal{A}'_V$ be the algebras of interacting field observables constructed via the two prescriptions, each of which is a subalgebra of $\mathcal{X}[\varepsilon, g_{1\rm t}, g_{2\rm t}, \dots]$, where $g_{it}$ are the 't Hooft coupling parameters related to the couplings $g_i$ in the interaction via formula (118). Let the interacting quantum fields in these algebras be $\varepsilon^{|\mathcal{O}|/2+T}\mathcal{O}_V$, respectively $\varepsilon^{|\mathcal{O}|/2+T}\mathcal{O}'_V$ ($T$ the number of traces), defined as formal power series in $\varepsilon$ and the 't Hooft parameters $g_{it}$.

**Proposition 3.** For any given $V = \sum g_i \mathcal{O}_i \in \mathcal{V}^{\rm inv}$ there exists a $V' = \sum g'_i \mathcal{O}_i \in \mathcal{V}^{\rm inv}$ and a *-isomorphism

$$R : \mathcal{A}_V \to \mathcal{A}'_{V'} \tag{123}$$

such that $g'_i = g'_{\rm it} \varepsilon^{T_i + |\mathcal{O}_i|/2 - 2}$ ($T_i$ is the number of traces in the field $\mathcal{O}_i$), with

$$g'_{\rm it} = g'_{\rm it}(g_{1\rm t}, g_{2\rm t}, \dots, \varepsilon) \tag{124}$$

a formal power series in $g_{it}$ and $\varepsilon$ (i.e., containing only positive powers of $\varepsilon$). The action of $R$ on a local field is given by

$$R : \underbrace{\varepsilon^{|\mathcal{O}_i|/2+T_i} \mathcal{O}_{iV}}_{\in \mathcal{A}_V} \to \sum_j \mathcal{Z}_{ij} \cdot \underbrace{\varepsilon^{|\mathcal{O}_j|/2+T_j} \mathcal{O}'_{jV'}}_{\in \mathcal{A}'_{V'}}, \tag{125}$$

where $\mathcal{Z}_{ij}$ are formal power series in $g_{1\rm t}, g_{2\rm t}, \dots$ and $\varepsilon$, and where $T_i$ is the number of traces in $\mathcal{O}_i$. [Recall that the $\varepsilon$-normalization factor in expressions like $\varepsilon^{|\mathcal{O}|/2+T}\mathcal{O}_V$ in the above equation is precisely the factor needed to make the latter an element of $\mathcal{A}_V$.] A similar formula holds for the time ordered products.

Moreover, if the "prime" prescription is related to the "unprime" prescription via a multiplicative change of scale (with multiplication factor $\lambda$), then each term in the expansion of $g'_{\rm it}(\lambda)$ and $\mathcal{Z}_{ij}(\lambda)$ depends at most polynomially on $\ln \lambda$, e.g.,

$$\mathcal{Z}_{ij}(\lambda) = \sum_{a_1, a_2, \dots, h \geq 0} z_{ij, a_1 a_2 \dots h}(\ln \lambda) g_{1\rm t}^{a_1} g_{2\rm t}^{a_2} \dots \varepsilon^h \tag{126}$$

where the $z_{ij,a_1 a_2 \dots h}$ are polynomials in $\ln \lambda$.

*Proof.* For any given, but fixed $N$, one can show by the same arguments as in [11] that any two prescriptions $T$ and $T'$ for defining time ordered products with properties (t1)–(t8) are related to each other in the following way:

$$T'\left(\prod_{i=1}^{n} f_i \mathcal{O}_i\right) = T\left(\prod_{i=1}^{n} f_i \mathcal{O}_i\right) + \sum_{\cup_j I_j = \{1,\dots,n\}} T\left(\prod_j \delta_{|I_j|}(\prod_{k \in I_j} f_k \mathcal{O}_k)\right). \tag{127}$$

Here, the following notation has been introduced: The sum runs over all partitions of the set $\{1, \dots, n\}$, excluding the trivial partition. The $\delta_k$ are maps

$$\delta_k : \otimes^k \mathscr{D}(\mathbb{R}^d; \mathcal{V}^{\text{inv}}) \to \mathscr{D}(\mathbb{R}^d; \mathcal{V}^{\text{inv}}), \tag{128}$$

characterizing the difference between $T$ and $T'$ at order $k$. The maps $\delta_k$ have the form[26]

$$\delta_k(\prod_{i=1}^{k} f_i \mathcal{O}_i) = \sum_i F_{k,i} \Psi_i, \tag{129}$$

where the functions $F_{k,i}$ are of the form

$$F_{k,i}(x) = \sum_{(\mu_1)\dots(\mu_k)} c_{k,i}{}^{(\mu_1)\dots(\mu_k)} \prod_k \partial_{(\mu_k)} f_k(x), \tag{130}$$

with each $(\mu_i)$ denoting a symmetrized spacetime multi index $(\mu_{i1} \dots \mu_{is})$, and with each $c_{k,i}{}^{(\mu_1)\dots(\mu_k)}$ denoting a Lorentz invariant tensor field (independent of $x$). Let us define a $V'$ in $\mathcal{V}^{\text{inv}}[g_1, g_2, \dots]$ (the space of formal power series in $g_i$ with coefficients in $\mathcal{V}^{\text{inv}}$) by

$$V' = \lim_{j \to \infty} \sum_{k \geq 1} \frac{i^k}{k!} \delta_k(\prod^k \theta_j V), \tag{131}$$

where $\{\theta_j\}$ represents any series of cutoff functions that are equal to 1 in compact sets $K_j$ exhausting $\mathbb{R}^d$ in the limit as $j$ goes to infinity. Then, for any given but fixed $N$, the result [11] establishes the existence of an isomorphism $R$ between the algebras of interacting field observables associated with the two prescriptions satisfying eq. (121) for some set of formal power series $Z_{ij}$ in $g_1, g_2, \dots$, where the fields in that equation are now given by gauge invariant expressions in $\mathcal{V}^{\text{inv}}$.

In order to prove the theorem, we must show that the interaction $V'$ and the factors $Z_{ij}$ appearing in the automorphism $R$ have the $N$-dependence specified by eqs. (121) respectively (120). This will guarantee that the above automorphisms $R$ defined separately

[26] Note that $\delta_1$ is not the identity, since we are allowing ambiguities in the definition of Wick powers, rather than defining them by normal ordering.

for each $N$ given rise to a corresponding automorphism of the interacting field algebras, viewed now as depending on $\varepsilon = 1/N$ as a free parameter.

In order to analyze the $N$-dependence of $V'$, let us consider the prescription $T''$ defined by $T'$ when applied to $k$ factors or more, and defined by eq. (127) when applied to $n \leq k-1$ factors. Then, by definition, the prescriptions $T$ and $T''$ will agree on $n \leq k-1$ factors, and

$$T''\left(\prod_{i=1}^{k} f_i \mathcal{O}_i\right) - T\left(\prod_{i=1}^{k} f_i \mathcal{O}_i\right) = \sum_i \Psi_i(F_{k,i}), \tag{132}$$

where $F_{k,i}$ is as in eq. (130). If $S_i$ is the number of traces in the field $\Psi_i$, then we claim that

$$\Psi_i(F_{k,i}) = O(1/N^{S_i+2k-2-\sum T_j+|\mathcal{O}_j|/2}), \tag{133}$$

where we recall that an algebra element $A \in \mathcal{W}_N^{\rm inv}$ given for all $N$ is said to be $O(1/N^h)$ if it can be written as $1/N^h$ times a sum of terms of the form $W_a(t_a)$, with each $t_a \in \mathcal{E}'_{|a|}$ depending only on positive powers of $1/N$. Using eq. (75), eq. (133) is equivalent to

$$F_{k,i} = O(1/N^{S_i+|\Psi_i|/2+2k-2-\sum T_j+|\mathcal{O}_j|/2}). \tag{134}$$

Assuming that this has been shown, we get the statement (120) about the $N$-dependence of $V'$ by plugging this relation into eqs. (128), (130) and (131), and using the definition of the ‘t Hooft couplings, eq. (118). In order to show (134), let us begin by introducing the “connected time ordered product” as the map $T^{\rm conn} : \otimes^k \mathscr{D}(\mathbb{R}^d, \mathcal{V}^{\rm inv}) \to \mathcal{W}_N^{\rm inv}$ defined recursively in terms of $T$ by

$$T^{\rm conn}\left(\prod_{i=1}^{n} f_i \mathcal{O}_i\right) \equiv T\left(\prod_{i=1}^{n} f_i \mathcal{O}_i\right) - \sum_{\{1,\dots,n\}=\cup I} \prod_{I}^{\rm class} T^{\rm conn}\left(\prod_{j \in I} f_j \mathcal{O}_j\right), \tag{135}$$

where the “classical product” $\cdot_{\rm class}$ is the commutative associative product on $\mathcal{W}_N^{\rm inv}$ defined by eq. (94), and where the trivial partition $I = \{1, \dots, k\}$ is excluded in the sum. By definition, we have $T''^{\rm conn} = T^{\rm conn}$ when acting on $n \leq k-1$ factors, because $T'' = T$ in that case. This implies that we can alternatively write $\sum_i \Psi_i(F_{k,i})$ in eq. (132) as the corresponding difference of connected time ordered products. By a line of arguments similar to the proof of eq. (104) in lemma 1 using that only connected Feynman diagrams contribute to the connected time ordered products, it can be seen that have

$$T^{\rm conn}\left(\prod_{i=1}^{k} f_i \mathcal{O}_i\right) = \sum_j (1/N)^{j+2k-2-\sum T_l+|\mathcal{O}_l|/2} \sum_{a=(a_1,\dots,a_j)} W_a(t_a(\otimes_i f_i)), \tag{136}$$

where each $t_a$ is a linear map

$$t_a : \otimes^k \mathscr{D}(\mathbb{R}^d) \to \mathcal{E}'_{|a|} \tag{137}$$

which can contain only positive powers of $1/N$. A completely analogous estimate holds for $T''^{\rm conn}$, with $t_a$ replaced by maps $t''_a$ with the same property. By eq. (132) (with the time ordered products replaced by the connected products in that equation), we therefore find

$$\sum_i \Psi_i(F_{k,i}) = \sum_j (1/N)^{j+2k-2-\sum T_l + |\mathcal{O}_l|/2} \sum_{a=(a_1,\dots,a_j)} W_a(s_a), \tag{138}$$

where we have set $s_a = t_a - t''_a$. We now write the expressions appearing on the left side as $\Psi_i(F_{k,i}) = W_a(u_a)$, where the distributions $u_a$ are related to $\Psi_i$ and $F_{k,i}$ via a relation of the form (28) and (29). If we now match the terms on both sides of this equation and use the linear independence of the $W_a$'s, we obtain the desired estimate (133). As already explained, this proves the desired $N$-dependence of $V'$.

The proof that the field strength renormalization factors $Z_{ij}$ in eq. (121) have the desired $N$-dependence expressed in eq. (125) is very similar to the proof that we have just given, so we only sketch the argument. For a given, but fixed $N$, the factors $Z_{ij}$ are defined implicitly by the relation

$$\lim_{l\to\infty} \sum_{k\geq 0} \frac{i^k}{k!} \delta_{k+1}(f\mathcal{O}_i, \prod^k \theta_l V) = \sum_j Z_{ij} f \mathcal{O}_j. \tag{139}$$

The desired $N$-dependence of the field strength renormalization factors implicit in eq. (125) is equivalent to

$$Z_{ij}(\varepsilon, g_1, g_2, \dots) = \varepsilon^{|\mathcal{O}_i|/2+T_i-|\mathcal{O}_j|/2-T_j} \cdot \mathcal{Z}_{ij}(\varepsilon, g_{1\rm t}, g_{2\rm t}, \dots), \tag{140}$$

where $\mathcal{Z}_{ij}$ are formal power series in the 't Hooft parameters and $\varepsilon$ (i.e., depending only on positive powers of $\varepsilon$), and where $T_i$ is the number of traces in the field $\mathcal{O}_i$. In order to prove this equation from the definition (139), one proceeds by analyzing the $N$-dependence of the maps $\delta_{k+1}$ in the same way as above.

The desired *polynomial* dependence of the coefficients of $\mathcal{Z}_{ij}(\lambda)$ and $g'_{i\rm t}(\lambda)$ on $\ln\lambda$ when $T'$ arises from $T$ via a scale transformation follows as in [10] from the almost homogeneous scaling behavior (21) of the time ordered prodcuts in the free theory. $\square$

# 6 Reduced symmetry

In the previous sections, we have constructed interacting field algebras associated with a $U(N)$-invariant action as a power series in $1/N$. Instead of considering $U(N)$-invariant actions, one can also consider actions that are only invariant under some subgroup. In the present section we will consider actions of the form eq. (81) in which the free part of the action is invariant under the full $U(N)$-group, and in which the interaction term $V$ is now invariant only under a subgroup $G$ of the form

$$G = U(N_1) \times \cdots \times U(N_k) \subset U(N), \tag{141}$$

where $\sum N_\alpha = N$. Since the perturbative construction of a quantum field theory with such an interaction involves Wick powers and time ordered products in the free theory that are not invariant under the full $U(N)$ symmetry group but only under the subgroup, we begin by describing the algebra

$$\mathcal{W}_N^G = \{A \in \mathcal{W}_N \mid \alpha_U(A) = A, \quad \forall U \in G\} \tag{142}$$

of observables invariant under $G$ of which these fields are elements. If we define $P_\alpha$ to be the projection matrix corresponding to the $\alpha$-th factor in the product (141), then it is easy to see that $\mathcal{W}_N^G$ is spanned by expressions of the form

$$W_{a,\alpha}(t) = \frac{1}{N^{|a|/2}} \int : \mathrm{Tr}\left(\prod_{i_1 \in I_1} \phi(x_{i_1}) P_{\alpha_{i_1}}\right) \cdots \mathrm{Tr}\left(\prod_{i_T \in I_T} \phi(x_{i_T}) P_{\alpha_{i_T}}\right) : t(x_1, \dots, x_{|a|}) \prod_j d^d x_j. \tag{143}$$

Here, $a$ represents a multi index $(a_1, \dots, a_T)$, $\alpha$ represents a multi index $(\alpha_1, \dots, \alpha_{|a|})$, the $I_j$'s are mutually disjoint index sets with $a_j$ elements each such that $\cup_j I_j = \{1, \dots, |a|\}$, and $t$ is a distribution in the space $\mathcal{E}'_{|a|}$.

The large $N$ limit of the algebras $\mathcal{W}_N^G$ can be taken in a similar way as in the case of full symmetry described in section 3, provided that the ratios

$$s_\alpha = N_\alpha / N \tag{144}$$

have a limit. As above, the large $N$ limit is incorporated in the construction of a suitable algebra $\mathcal{X}_{\underline{s}}[\varepsilon]$, depending now on the ratios $\underline{s} = (s_1, \dots, s_k)$, of polynomial espressions in $\varepsilon$ whose coefficients are given by generators $W_{a,\alpha}(t)$. To work out the algebra product between two such generators as a power series in $\varepsilon = 1/N$, it is useful again to consider first the simplest case $d = 0$ corresponding to the matrix model given by the action functional (47), and by considering the product of the matrix generators

$$W_{a,\alpha} = \frac{1}{N^{|a|/2}} : \prod_i^T \mathrm{Tr}\left(\underbrace{M P_{\alpha_i} M P_{\alpha_{i+1}} \cdots M P_{\alpha_k}}_{a_i \text{ factors of } M}\right) :, \tag{145}$$

corresponding to the algebra elements (143). We expand the product of two such generators in terms of Feynman graphs as in section 3, the only difference being that the vertices corresponding to the traces in eq. (145) now also contain projection operators $P_\alpha$. We take this into account by modifying our notation of these vertices by indicating also the projectors adjacent to the vertex. As an example, consider the generator with a single trace given by

$$W_{3,(\alpha_1,\alpha_2,\alpha_3)} = \frac{1}{N^{3/2}} : \mathrm{Tr}\, P_{\alpha_1} M P_{\alpha_2} M P_{\alpha_3} M : . \tag{146}$$

This generator will contribute a 3-valent vertex drawn in the following picture:

A closed loop of index contractions occurring in a diagram associated with the product $W_{a,\alpha} \cdot W_{b,\beta}$ will contribute a factor of

$$\operatorname{Tr}\left(\prod_{\gamma\in\{\alpha_i,\beta_j\}} P_\gamma\right), \tag{147}$$

where the product is over all projectors that are encountered when following the index loop. But the projection matrices $P_\gamma$ are mutually orthogonal, $P_{\gamma'}P_\gamma = \delta_{\gamma\gamma'}P_\gamma$, so this factor is given by $N_\gamma$ if the projectors in the index loop are all equal to some $\gamma$, and vanishes otherwise. If we let $I_\alpha$ be the number of index loops in a given graph containing only projections on the $N_\alpha$-subspace, then we consequently get

$$W_{a,\alpha} \cdot W_{b,\beta} = \sum_{\text{graphs}} s_1^{I_1} \dots s_k^{I_k} \varepsilon^{J+\sum H_k+\sum(V_k-2)} \prod_{\text{lines } (k,l)} \frac{1}{m^2} \cdot W_{c,\gamma}, \tag{148}$$

where the sum is only over graphs whose index loops contain only one kind of projectors, and where $\varepsilon = 1/N$ as usual. As in the previous section, $W_{\gamma,c}$ arises from the graphs with loops containing $c_j$ external currents each, and $c$ denotes the multi index $(c_1, c_2, \dots)$. The new feature is that each of these loops now also contains projection operators $P_\gamma$.

As in the previous section, we can generalize the considerations leading to formula (148) to determine the product of algebra elements $W_{a,\alpha}(t), t \in \mathcal{E}'_{|a|}$ when the spacetime dimension is not zero. We thereby obtain a family of algebras $\mathcal{X}_{\underline{s}}[\varepsilon]$ depending analytically on the ratios $\underline{s} = (s_1, \dots, s_k)$. Since by definition $0 \le s_\alpha \le 1$ and $\sum s_\alpha = 1$, the tuples $\underline{s}$ can naturally be viewed as elements of the standard $(k-1)$-dimensional simplex

$$\Delta_{k-1} = \{(s_1, \dots, s_k) \in \mathbb{R}^k \mid 0 \le s_\alpha \le 1, \sum s_\alpha = 1\}. \tag{149}$$

Thus, our construction of the algebras associated with the reduced symmetry group yields a bundle

$$\sigma_{k-1} : \Delta_{k-1} \to \mathrm{Alg}, \quad \underline{s} \to \mathcal{X}_{\underline{s}}[\varepsilon] \tag{150}$$

of algebras with every point of the standard $(k-1)$-symplex for every $k$. The parameters $\underline{s}$ interpolate continuously between situations of different symmetry. If $\Delta_l$ is a face of $\Delta_k$, (so that $l < k$), then the assignment fulfills the "self-similar" restriction property

$$\sigma_k \restriction \Delta_l = \sigma_l. \tag{151}$$

The extremal points of $\Delta_k$ (i.e., the zero-dimensional faces) correspond to full symmetry, i.e., the restriction of $\sigma_k$ to these points yields the algebras $\mathcal{X}[\varepsilon]$ constructed in section 3.

We now repeat the construction of the interacting field algebras $\mathcal{A}_V$, $V = \sum g_i \mathcal{O}_i$, with each $\mathcal{O}_i$ an expression in the field that is invariant under the reduced symmetry group. We denote the vector space of such formal expressions by

$$\mathcal{V}_k^{\rm inv} = {\rm span}\left\{ \mathcal{O} = \prod_i {\rm Tr}\left(\prod^{a_i}(P_{\alpha_l}\partial_{\mu_1}\cdots\partial_{\mu_j}\phi)\right), \alpha_l = 1,\dots,k \right\}, \tag{152}$$

(note that $\mathcal{V}_1^{\rm inv}$ can naturally be identified with $\mathcal{V}^{\rm inv}$ in the notation introduced earlier). For an arbitrary but fixed $N$, a gauge invariant interaction $V = \sum g_i \mathcal{O}_i \in \mathcal{V}_k^{\rm inv}$ gives rise to corresponding interacting quantum fields $\mathcal{O}_V$ and their time ordered products as elements in the corresponding algebra $\mathcal{W}_N^G[g_1, g_2, \dots]$. The limit $N \to \infty$ can be taken on the algebraic level in the same way as in the case of full symmetry described in section 4, provided that the ratios $s_\alpha = N_\alpha/N$ are held fixed, and provided that the couplings are tuned as in (118). This construction directly leads to an algebra $\mathcal{A}_{V,\underline{s}}$ of formal power series in $\varepsilon = 1/N$ as well as the 't Hooft couplings $g_{i\rm t}$ of which the smeared normalized gauge invariant interacting fields $\varepsilon^{|\mathcal{O}|/2+T}\mathcal{O}_V(f)$ and their time ordered products are elements. The algebra $\mathcal{A}_{V,\underline{s}}$ is now a subalgebra of the algebra $\mathcal{X}_{\underline{s}}[\varepsilon, g_{1\rm t}, g_{2\rm t}, \dots]$ of formal power series in the 't Hooft couplings, with coefficients in the algebra $\mathcal{X}_{\underline{s}}[\varepsilon]$.

We have constructed in this way a family algebra $\mathcal{A}_{V,\underline{s}}$ parametrized by deformation parameters $\underline{s}$, i.e., a bundle

$$\sigma_{V,k} : \Delta_{k-1} \to {\rm Alg}, \quad \underline{s} \to \mathcal{A}_{V,\underline{s}}, \tag{153}$$

where $\Delta_{k-1}$ is the $(k-1)$-dimensional standard simplex (149) of which the $\underline{s}$ are elements. These deformation parameters smoothly interpolate between situations of different symmetry as well as between different interactions. For example, $s_1 = 1, s_2 = \cdots = s_k = 0$ (i.e., $N_1 = N$) corresponds to the extremal case of full symmetry, where only those terms in the interaction $V \in \mathcal{V}_k^{\rm inv}$ contribute that contain only projectors $P_1$ associated with the $N_1$-factor in the symmetry group. More generally, if $\Delta_l$ is a face of $\Delta_k$, $l < k$ then we have

$$\sigma_{V,k} \restriction \Delta_l = \sigma_{V,l}, \tag{154}$$

where it is understood that the "$V$" appearing in $\sigma_{V,l}$ is the formal expression in $\mathcal{V}_l^{\rm inv}$ obtained by dropping in $V \in \mathcal{V}_k^{\rm inv}$ all terms containing projectors that are not associated

with the extremal points of $\Delta_l$. Using the restriction property (154), one can also construct bundles of interacting field algebras over an arbitrary $k$-dimensional ($C^0$-) manifold $X$ by triangulating $X$ into simplices $\Delta_l$.

As in the case of full symmetry, the perturbative expansion of the interacting fields $\varepsilon^{|\mathcal{O}|/2+T}\mathcal{O}_V$ defined via an interaction $V \in \mathcal{V}_k^{\text{inv}}$ as an element of $\mathcal{A}_{V,\underline{s}}$, $\underline{s} = (s_1, \dots, s_k)$, is organized in terms of Feynman graphs that are associated with Riemann surfaces. Moreover, the faces of these Feynman graphs defined by the closed index loops are now "colored" by the numbers $s_\alpha$. To illustrate this in an example, consider the interacting field $\varepsilon^{3/2}(\operatorname{Tr}\phi)_{\theta V}$ in case of 3 colors, $k = 3$, with interaction $\theta(x)V$, where $\theta$ is a cutoff function, and where we take $V$ to be

$$V(\phi) = g \sum_{\alpha_i \in \{1,2,3\}} \operatorname{Tr}(P_{\alpha_1}\phi P_{\alpha_2}\phi \cdots P_{\alpha_n}\phi), \tag{155}$$

In order to make things a little more interesting, we restrict the sum in this equation to sequences of colors $(\alpha_1, \dots, \alpha_n)$ such that

$$\alpha_1 \neq \alpha_2 \cdots \neq \alpha_n \neq \alpha_1. \tag{156}$$

In the graphical notation introduced above this condition means that the vertices occurring in $V$ are restricted by the property that adjacent projectors $P_{\alpha_i}$ (as one moves around the vertex) are different.

A retarded product appearing in the perturbative expansion of $\varepsilon^{3/2}(\operatorname{Tr}\phi)_{\theta V}$ with $V$ given by eq. (155), can now be written as a sum of contributions from individual Feynman graphs as follows:

$$\begin{aligned}\varepsilon^{3/2} R(V(y_1)\cdots V(y_n); \operatorname{Tr}\phi(x)) = g_{\text{t}}^n \sum_{\text{genera } H} \varepsilon^H \sum_{\text{graphs } \Gamma} \sum_{\text{colorings } \mathcal{C}} C_{\mathcal{C},\Gamma} \cdot s_1^{F_1} s_2^{F_2} s_3^{F_3} \cdot \\ \cdot\, \varepsilon^{f/2+T} r_\Gamma(y_1, \dots, y_n; x) \;\; : \operatorname{Tr} \prod_i \prod_{\alpha_i} \phi(y_i) P_{\alpha_i} \cdots \operatorname{Tr} \prod_j \prod_{\beta_j} \phi(y_j) P_{\beta_j} :_H .\end{aligned} \tag{157}$$

The notation used in the expression is analogous to that in the corresponding equation (120) in the case of full symmetry, with the following differences: The $r_\Gamma$ are the distributions appearing in the expansion of the interacting field in scalar $\phi^n$-theory. By contrast to eq. (120), there appears now an additional sum over colorings, $\mathcal{C}$, over all ways to assign colors $s_1, s_2, s_3$ to those little surfaces in the big surface $\mathscr{S}$ associated with the Feynman graph not containing currents, in such a way that adjacent surfaces are never occupied by the same color (this corresponds to the property (156) of $V$), and $F_\alpha$ is the number of such little surfaces colored by $s_\alpha$. The combinatorical factor $C_{\Gamma,\mathcal{C}}$ counts the number of ways in which a given coloring scheme can be produced by assigning the different terms in $V$ to the vertices. The external currents are again collected in the normal ordered term appearing in eq. (157), where each trace corresponds to following through

the index line to which the currents within that trace belong, but there now appear also the projectors $P_\alpha$ that are encountered when following through such an index line. A similar expansion can be written down for the interacting fields without cutoff $\theta$, as defined in eq. (87).

Thus, roughly speaking, the Feynman expansion of on interacting field $\varepsilon^{3/2}(\mathrm{Tr}\,\phi)_V$ with interaction $V$ given by (155) differs from the corresponding expansion of $\phi_V$ in the theory of a single scalar field with $V = g\phi^n$ only in that the little surfaces in the graphs defined by the propagator lines are now colored according to the structure of the interaction $V$, and each coloring is weighted by the number $\prod_\alpha s_\alpha^{F_\alpha}$ where $F_\alpha$ is the number of little surfaces collored by $\alpha \in \{1, 2, 3\}$. The property (156) of the interaction chosen in our example implies that only those graphs occur which can be colored by 3 colors in such a way that adjacent little surfaces have different colors. In other words, there cannot appear any Feynman graphs such that the associated surface cannot be colored by less than 4 colors in this way. Thus, by choosing the interaction $V$ in the way described above, we have, in effect, suppressed certain Feynman graphs that would be present in scalar $\phi^n$-theory.

# 7 Summary and comparison to other approaches

In this paper, we have constructed perturbatively the gauge invariant interacting quantum field operators for scalar field theory in the adjoint representation of $U(N)$, with an arbitrary gauge invariant interaction. These operators are members of an abstract algebra, whose structure constants, as we demonstrated, have a well-defined limit as $N \to \infty$, or, more properly, are power series in $1/N$, with positive powers (provided the coupling parameters are also rescaled in a specific way by suitable powers of $1/N$). In this sense, these algebras, and the interacting quantum fields that are the elements of this algebra, also possess a well-defined large $N$ limit. We showed that the renormalization group flow can be defined on the algebraic level via a 1-parameter family of isomorphisms acting on the fields via a rescaling of the spacetime arguments, the field strength, and an appropriate change in the coupling parameters. That flow was shown to be a power series in $1/N$ with positive powers, and hence has a large $N$ limit. We also presented similar results in the case when the interaction of the fields is not invariant under $U(N)$, but only invariant under certain diagonal subgroups. We did not address issues related to the convergence of the perturbation expansion or the expansion in $1/N$.

Our motivation for investigating the formulation of the $1/N$-expansion in an algebraic framework rather than via Green's functions of the vacuum state — as is conventionally done — was that the algebraic formulation is completely local in nature and thereby bypasses potential infra-red problems, which can occur in the usual formulations via Green's functions in massless theories. Also, although we explicitly only worked in Minkowski space, we were strongly motivated by the fact that an algebraic approach is essential if one wants to formulate quantum field theory in a generic curved spacetime, where no pre-

ferred vacuum state exists. Actually, since our arguments are mostly of combinatorical nature, we expect that the present algebraic formulation of the $1/N$ expansion can be carried over rather straightforwardly to curved space.

The algebraic approach presented in this paper is rather different in appearance from the usual formulation via Green's functions, so we would briefly like to explain the relationship between the two approaches. In the conventional approach, one considers the $N$-dependence of the vacuum Green's functions[27] of gauge invariant interacting fields, $G_n = \omega_0(\mathcal{O}_V \dots \mathcal{O}_V)$ associated with the interaction $V$. For example, for $\mathcal{O} = \mathrm{Tr}\,\phi^2$, one finds that the corresponding connected Green's function $G_n^{\mathrm{conn}}$ receives contributions of order $N^{2-2H}$ from Feynman graphs of genus $H$ (assuming that the couplings in $V$ are scaled by appropriate powers of $1/N$). Thus, the planar diagrams $H = 0$ make the leading contribution at large $N$, with $G_n^{\mathrm{conn}} \sim N^2$, independent of $n$.

If one wants to reconstruct from the Green's functions the Hilbert space of the theory and the interacting field observables as linear operators on that Hilbert space, one needs to consider not the connected Green's functions, but the Wightman Green's functions $G_n$ themselves, since the latter enter in the Wightman reconstruction argument. Writing the Wightman Green's functions $G_n$ in terms of $G_n^{\mathrm{conn}}$ via the usual formulae, one immediately gets that $G_n \sim N^{2n}$. Hence, it is clear that, if one wants these constructions to be well defined at infinite $N$, then one needs to consider the normalized fields $N^{-2}\mathrm{Tr}\,\phi^2$. Similar remarks also apply to more general composite fields, with appropriate powers of $1/N$ in the normalization factor, depending on the number of traces and the number of basic fields. These powers coincide precisely with the powers found in our algebraic approach (see eq. (114)) by different means. The arguments that we have just given are of course only formal, because the reconstruction theorem, as it stands, is not really applicable in perturbation theory. Also, as we have already emphasized several times, the $G_n$ may actually be ill defined because they may involve infra-red divergent integrations over interaction vertices (in massless theories). The methods of this paper, on the other hand, give a rigorous construction of the field theory at the algebraic level that, by contrast to the formulation via Green's functions, should also be applicable in curved spacetimes.

It is a trivial consequence of the large $N$-behavior of the connected Green's functions that, in the large $N$ limit, the Wightman Green's functions $\hat{G}_n$ of the suitably normalized field operators factorize into 1-point functions, $\hat{G}_n(x_1, \dots, x_n) \sim \hat{G}_1(x_1) \cdots \hat{G}_1(x_n)$. Thus, it is formally clear that the large $N$ theory is *abelian*, i.e., the field commutators vanish. This can be seen explicitly in our algebraic framework, since the commutator of any two (suitably normalized) interacting fields is seen to be of order $N^{-2}$. Thus, the algebra $\mathcal{A}_V$ of interacting fields is abelian in the large $N$ limit, and consequently the representations are degenerate. This behavior can be nicely formalized in the algebraic framework by viewing

[27] One normally considers time ordered Green's functions, but the arguments do not depend on the time ordering and therefore also equally apply to the Wightman functions, which we prefer to consider here.

$\mathcal{A}_V$ as a *Poisson algebra*[28], with antisymmetric bracket defined by $\{\,.\,,\,.\,\} = N^2[\,.\,,\,.\,]$. In the limit of large $N$, that Poisson algebra becomes abelian, as is also the case for the classical limit[29] $\hbar \to 0$ (the appropriate definition of the Poisson bracket in that case being $\{\,.\,,\,.\,\} = (i\hbar)^{-1}[\,.\,,\,.\,]$). Thus, it is seen clearly at the algebraic level that there exist formal similarities between the large $N$ limit and the classical limit, and that, in particular, the large $N$ limit of a field theory does not define a quantum field theory in the usual sense, but rather a Poisson algebra. We note that the large $N$ limit can thereby be interpreted, within our algebraic framework, as some kind of "deformation quantization" [1], the deformation parameter being $1/N^2$.

**Acknowledgements:** I would like to thank K. H. Rehren for useful conversations. This work was supported by NSF-grant PH00-90138 to the University of Chicago.

# References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer: "Deformation Theory and Quantization," Annals of Physics (N.Y.) **111**, 61 (1978)

[2] R. Brunetti, K. Fredenhagen and M. Köhler: "The microlocal spectrum condition and Wick polynomials on curved spacetimes," Commun. Math. Phys. **180**, 633-652 (1996)

[3] R. Brunetti, K. Fredenhagen: "Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on physical backgrounds," Commun. Math. Phys. **208**, 623-661 (2000)

[4] M. Dütsch and K. Fredenhagen: "Algebraic quantum field theory, perturbation theory, and the loop expansion," Commun. Math. Phys. **219** (2001) 5, [arXiv:hep-th/0001129]; "Perturbative algebraic field theory, and deformation quantization," to appear in: Fields Inst. Commun., [arXiv:hep-th/0101079]

[28] A Poisson algebra is an algebra $\mathcal{A}$ together with an antisymmetric bracket $\{\,.\,,\,.\,\}$ from $\mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$ satisfying the Leibniz rule $\{ab, c\} = a\{b, c\} + \{a, c\}b$, together with the Jacobi identity. A Poisson algebra is called abelian if $\mathcal{A}$ is abelian. The observables of a classical field theory form an abelian Poisson algebra, with the (commutative) algebra multiplication given by pointwise multiplication of the observables, and with the Poisson bracket given in terms of the symplectic structure of the theory. A trivial example of a nonabelian Poisson algebra is any non-commutative algebra with the Poisson bracket defined by the algebra commutator.

[29] The classical limit and the large $N$ limit are not, of course, equivalent.

[5] M. Dütsch and K. Fredenhagen, “The master Ward identity and generalized Schwinger-Dyson equation in classical field theory,” Commun. Math. Phys. **243**, 275 (2003), [arXiv:hep-th/0211242].

[6] M. Dütsch and K. Fredenhagen: “Causal Perturbation Theory in Terms of Retarded Products, and a Proof of the Action Ward Identity,” [hep-th/0403213]

[7] M. Dütsch, T. Hurth and G. Scharf, “Causal Construction Of Yang-Mills Theories. 4. Unitarity,” Nuovo Cim. A **108**, 737 (1995). See also references contained in this paper.

[8] R. Haag: “On quantum field theories,” Dan. Mat. Fys. Medd. **29**, 13 (1955) no. 12, reprinted in: Dispersion Relations and the Abstract Approach to Field Theory, L. Klein (ed.), Gordon & Breach, NY, 1961.

[9] L. Hörmander: “The Analysis of Linear Partial Differential Operators I,” 2nd Edition, Springer-Verlag (1990)

[10] S. Hollands and R. M. Wald, “On the renormalization group in curved spacetime,” Commun. Math. Phys. **237**, 123-160 (2003) [arXiv:gr-qc/0209029].

[11] S. Hollands and R. M. Wald, “Local Wick polynomials and time ordered products of quantum fields in curved spacetime,” Commun. Math. Phys. **223** (2001) 289 [arXiv:gr-qc/0103074].

[12] S. Hollands and R. M. Wald, “Existence of local covariant time ordered products of quantum fields in curved spacetime,” Commun. Math. Phys. **231**, 309 (2002) [arXiv:gr-qc/0111108].

[13] S. Hollands and R. M. Wald, “Time ordered products of fields with derivatives,” in preparation.

[14] S. Hollands and W. Ruan, “The state space of perturbative quantum field theory in curved space-times,” Ann. Inst. H. Poincare **3** (2002) 635 [arXiv:gr-qc/0108032].

[15] G. 't Hooft, “A Planar Diagram Theory For Strong Interactions,” Nucl. Phys. B **72**, 461 (1974).

[16] T. Hurth and K. Skenderis, “Quantum Noether method,” Nucl. Phys. B **541**, 566 (1999) [arXiv:hep-th/9803030].

[17] G. Parisi, “The theory of non-renormalizable interactions. The large $N$ expansion,” Nucl. Phys. **B100**, 368-388 (1975)

[18] M. Reed and B. Simon: "Methods of modern mathematical physics I," Academic Press, New York (1973)

[19] G. Scharf, "Finite Quantum Electrodynamics. The Causal Approach," 2nd edition, Springer Verlag (1995);